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The Book of Integers

MATHEMATICS

FOR HIGH SCHOOL STUDENTS

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The Book of Integers - Mathematics for High School Students

ISBN 0-9781703-1-8

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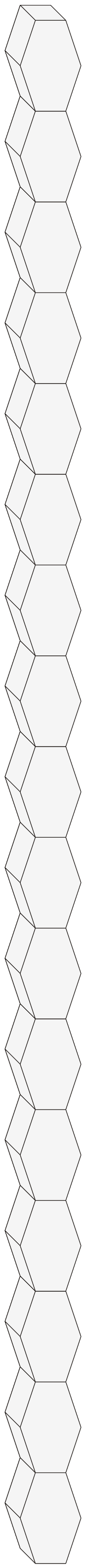


"Simplicity is complexity resolved."

"Don't look for obscure formulae or mysteries.
It is pure joy that I am giving you."

CONSTANTIN BRANCUSI, ROMANIAN SCULPTOR
(see on the left side a drawing of "The Endless Column")

Acknowledgements

We want to thank to all our students. Their feedback, comments, criticism, ideas, and requests help us everyday to become better teachers.

Iulia & Teodoru Gugoiu

Content

1. Understanding Integers

1.1 *(Definition)* Integers are *signed numbers* made of:
a) a *sign*, **+** for *positive numbers* and **-** for *negative numbers*
b) an *absolute value*, that is a *natural number*
Example: *-5*.
The sign is - . The absolute value is *5*.

$$3, \quad 1, \quad 13, \quad 25$$

1.2 *(No Sign Rule)* If an integer does not have a sign, by default the sign is considered positive. So, the numbers 5 and +5 are identical.
In general:

$$a \quad a$$

1.3 *(Number Zero) 0* is considered *neither positive nor negative*. However, some generalizations require us to consider the following numbers: *+0* and *-0*. Still, these two instances of the number zero are equal to *0*. So:

$$0 \quad 0 \quad 0$$

1.4 *(The Set of Integers)* The set of all integers can be represented by using the *set notation* as:

$$Z \quad \{..., 3, 2, 1, 0, 1, 2, 3, ...\}$$

The set of *positive integers* is:

$$Z \quad \{1, 2, 3, ...\}$$

1.5 *(Boundless)* The set of integers is *unlimited* or *boundless*. There is neither a biggest integer nor a smallest integer. The number of elements in this set (the number of integers) is *infinite*.

1.6 *(Infinity)* Although a largest integer does not exist as a regular number, the symbol (called *plus infinity*) is used to express an unlimited large positive number.
Similarly, the symbol (called *minus infinity*) is used to express an unlimited large (as absolute value) negative number.
These two symbols *are not considered numbers* because the arithmetic of these numbers is completely different from the arithmetic of regular numbers. For example:

$$5$$

is not defined

1. Identify whether the following numbers are integers or not:

$a)\ 1 \quad b)\ \sqrt{2} \quad c)\ 0 \quad d)\ 7.5 \quad e)\ 5 \quad f)\ 1 \quad g)\ \frac{2}{3} \quad h)\ 3 \quad i)\ 0.01 \quad j)\ 10 \quad k)$

2. For each of the following integers, identify the sign and the absolute value:

$a)\ 3 \quad b)\ 2 \quad c)\ 0 \quad d)\ 2 \quad e)\ 11 \quad f)\ 10 \quad g)\ 4 \quad h)\ 3 \quad i)\ 22 \quad j)\ 101$

3. Use the short notation (by dropping the sign) to rewrite the following integers:

$a)\ 5 \quad b)\ 2 \quad c)\ 1 \quad d)\ 0 \quad e)\ 3$

4. Rewrite the following numbers as integers (including the sign):

$a)\ 1 \quad b)\ 20 \quad c)\ 300 \quad d)\ 41 \quad e)\ 55$

5. Classify the following numbers as positive, negative or neither:

$a)\ 1 \quad b)\ 5 \quad c)\ 2 \quad d)\ 10 \quad e)\ 0 \quad f)\ 5 \quad g)\ 7 \quad h)\ 11 \quad i)\ 123 \quad j)\ 1000$

6. Find a way to represent the set of negative integers (see 1.4 for the set of positive integers).

7. Use the set notation (see 1.4) to represent all odd positive integers greater than 0 and less than 10.

8. Find the logical value (true or false) of each of the following sentences:

a) All integers are positive.
b) There is a largest positive integer.
c) The smallest integer is 0.
d) Any natural number is also an integer.
e) There are more positive integers than negative integers.
f) is a positive integer.

9. For each case, identify the integers that satisfy the given properties:

a) is positive and has an absolute value equal to 7
b) is negative and has an absolute value equal to 11
c) has an absolute value equal to 5
d) has an absolute value equal to 0
e) is positive and has an absolute value equal to

10. Find the value of the following expressions that contain the infinity symbol:

$a)\ 1 \quad b) \quad c)\ 10 \quad d) \quad e)\ /$

$f)\ ^{3} \quad g)\ \frac{}{1000} \quad h)\ 10 \quad i)\ \frac{1}{} \quad j)\ \frac{2}{}$

2. Absolute value, sign, and opposite

2.1 *(Absolute Value)* If you drop the sign of an integer, you get its *absolute value*. The absolute value of an integer is a natural number. Use the function | | to express the absolute value. Examples:

| 4| 4

(Read: the absolute value of *-4* is *4*)

| 6| 6

|7| | 7| 7

2.2 *(Number Zero)* The absolute value of *0* is *0*.

|0| 0

2.3 *(Sign)* If you drop the absolute value of an integer you get its sign. Use the function *sign()* to express the sign of an integer. Examples:

sign(5)

sign(7)

sign(10) *sign*(10)

2.4 *(Number Zero)* The sign of *0* is not defined. *0* is neither a positive nor a negative number.

2.5 *(Opposite Numbers)* Two integers are called *opposite* if they have the *same absolute value* but *opposite signs*.
So, *+5* and *-5* are opposite.
+5 is the opposite of *-5*
-5 is the opposite of *+5*

2.6 *(General Definition)* The opposite of any number *a* is *-a*. So, to get the opposite of a number, just place the - sign in front of the number. Examples:
The opposite of *5* is *-5*.
The opposite of *-5* is *-(-5)* that is equal to *5*.

2.7 *(The Opposite of an Opposite)* The opposite of an opposite of a number is equal to the number itself:

(*a*) *a*

Example. Let's consider the number *3*. The opposite of this number is *-3*. The opposite of the number *-3* is *+3*, which is equal to the original number *3*.

2.8 *(The Opposite of Zero)* The opposite of *0* is *0*:

0 0 0

1. Use absolute value notation (see 2.1), and find the absolute value of the following integers:

a) 3 *b*) 2 *c*) 2 *d*) 3 *e*) 0 *f*) 10 *g*) 100 *h*) 12 *i*) 7 *j*) 7

2. For each of the following absolute values, find the possible value(s) of the integer(s):

a) 0 *b*) 1 *c*) 2 *d*) 3 *e*) 10 *f*) 5 *g*) 123

3. For each case, find the sign of the integer (see 2.3):

a) 3 *b*) 5 *c*) 4 *d*) 0 *e*) 10 *f*) 100 *g*) 10 *h*) 1 *i*) 1 *j*) 7

4. For each case, find the opposite of the given integer:

a) 12 *b*) 20 *c*) 3 *d*) 3 *e*) 20 *f*) 0 *g*) 11 *h*) 2 *i*) 7 *j*) 77

5. Find the value of each expression. One case is solved for you as an example:

a) (2) *b*) (5) *c*) (3) *d*) ((4))) *e*) ((20)))
f) (1 2) *g*) (11 8) *h*) (5 4 7) *i*) ((5 3)) *j*) (((2 3)))

Example:

j) (((2 3))) (((5))) ((5)) (5) 5

6. Find the value of each expression that contains the absolute value function. One case is solved for you as an example:

a) | (1 4)| *b*) |5 2| *c*) | (4)| *d*) | 3| *e*) | (7)|

f) | 3| | 1| *g*) | 5| | 2| *h*) 2 | 2| *i*) | 5| | 3| *j*) | 8| | 2|

Example:

g) | 5| | 2| 5 2 3

7. Find the value of each expression that contains the absolute value function. One case is solved for you as an example:

a) | 1| | 2| | 3| *b*) $\frac{| \ 10|}{| \ 5|}$ *c*) (| 5| 1) *d*) |3 1| | 5| *e*) 5 | 2|

f) 2 (| 3| 1) *g*) | 3| 2 *h*) 1 | 2| 3 *i*) | 5| | 3| *j*) || 7| | 3||

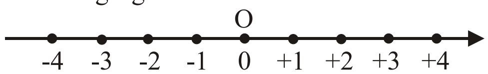
k) | (7 4)| *l*) | (2)| *m*) |5 |5 5|| *n*) | | 8|| *o*) | (1 | 2|)|

Example:

j) || 7| | 3|| |7 3| |4| 4

3. Number Line

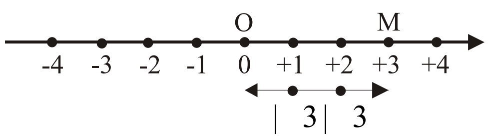
3.1 *(The Number Line)* An intuitive representation of integers is based on the *number line*. There is *one to one correspondence* between the set of integers and a set of *equidistant points* on a line. This correspondence is illustrated in the following figure:



3.2 *(Positive and Negative Integers)* The positive integers are situated to the *right side* of the origin *O*. The negative integers are situated to the *left side* of the origin *O*.

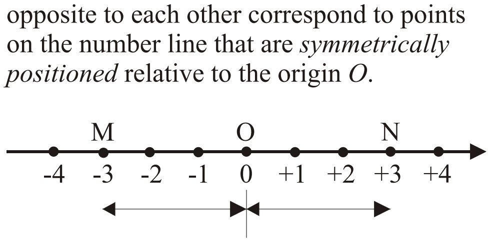
3.3 *(Number Zero)* Number zero corresponds to the origin *O*.

3.4 *(Absolute value)* The absolute value of an integer is equal to the *distance* between the origin *O* and the corresponding point on the number line. The greater the absolute value of an integer, the greater the distance between the corresponding point on the number line and the origin.



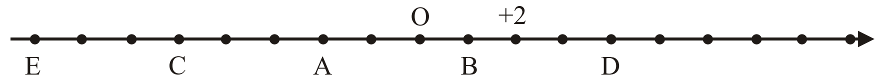
Example. The absolute value of the number *+3* is *3*. *M* is the point corresponding with *+3* on the number line. The distance between *M* and *O* is equal to *3*.

3.5 *(Opposites)* Two integers that are opposite to each other correspond to points on the number line that are *symmetrically positioned* relative to the origin *O*.



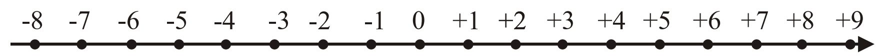
Example. The numbers *-3* and *+3* are opposite to each other. *-3* corresponds to point *M* on the number line and *+3* corresponds to *N*. The points *M* and *N* are symmetrically positioned relative to the origin *O*. The distances *MO* and *NO* are both equal to 3.

1. For each point represented on the following number line, find the corresponding integer:

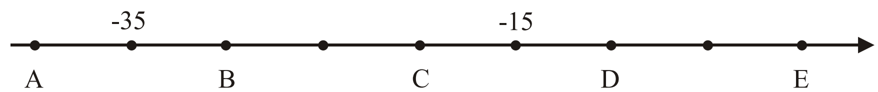


2. Plot the following points on the number line below:

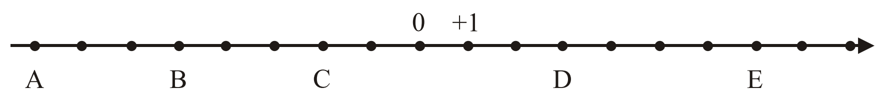
a) *A*(2) *b)* *B*(7) *c)* *C*(5) *d)* *D*(4) *e)* *E*(3)



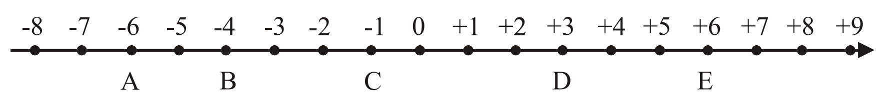
3. A part of a number line is represented in the figure below. All the points are equidistant. Find the integers that correspond to the given points:



4. Find the corresponding integer and its absolute value, by calculating the distance between each point and the origin O.

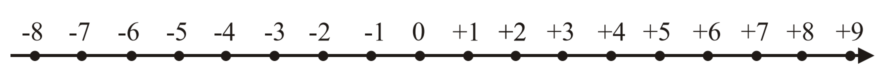


5. For each of the points below (A, B, C, D, E, F), find the symmetrical point relative to the origin O (see 3.5):



6. For each case, use the number line to identify an integer that is:

a) 2 units left of +7
b) 3 units right of +2
c) 2 units left of -5
d) 4 units right of -2
e) 3 units from +5
f) 2 units from -6
g) 3 units from the origin



4. Comparing Integers

4.1 *(Less, Equal, Greater)* Any two integers *a* and *b* can be *compared*. The result of this *comparison* is that only one of the following statements is true:

a b a is less that b

a b a is equalt to b

a b a is greater that b

4.2 *(The Number Line)* On the number line a smaller integer is placed at the left side of a bigger integer. So:

... 3 2 1 0 1 2 3 ...

... 3 2 1 0 1 2 3 ...

4.3 *(Order)* As you can see from the previous relation, the set of integers is an *ordered set*. Integers can be *sorted* (ordered) from smaller integers to bigger integers. There is neither a smallest integer nor a largest integer.

4.4 *(Comparing negative integers, 0, and positive integers)* Any negative integer is less than *0* and *0* is less than any positive integer. Examples:

2 0 0 2

0 7 7 0

4.5 *(Transitivity)* The order (comparison) relation between two integers has the *transitivity property*:

if a b and b c then a c

Example:

2 0 *and* 0 3 *then* 2 3

So, any negative integer is less than any positive integer.

4.6 *(Comparing positive integers)* When you compare two positive integers, the greatest integer is the one with the greatest absolute value. Example:

3 5 *or* 5 3

4.7 *(Comparing negative integers)* When you compare two negative integers, the greatest integer is the one with the smallest absolute value. Example:

7 5 *or* 5 7

Indeed, *-7* is at the left side of *-5* and therefore is smaller.

1. For each case identify the true statement:

a) 1 2; 1 2; 1 2 *b)* 5 3; 5 3; 5 3 *c)* 0 5; 0 5; 0 5

d) 3 3; 3 3; 3 3 *e)* 1 0; 0 1; 1 0 *f)* 0 0; 0 0; 0 0

2. Sort the integers from the smallest to the greatest (use the < symbol):

a) 2; 1 *b)* 0; 1; 2 *c)* 3; 1; 2; 0

d) 3; 1; 2; 0; 2 *e)* 1; 1; 2; 2; 3; 3 *f)* 3; 2; 0; 1; 2; 3

3. Sort the integers from the greatest to the smallest (use the > symbol):

a) 2; 1 *b)* 0; 1; 2 *c)* 3; 2; 1; 0

d) 3; 2; 2; 0; 4 *e)* 2; 2; 4; 4; 5; 5 *f)* 3; 2; 1; 1; 5

4. Use the transitivity property to combine the two true statements and get a third true statement (see 4.4):

a) 0 1 *and* 1 2 *b)* 2 0 *and* 0 2 *c)* 3 2 *and* 2 1

d) 5 2 *and* 2 4 *e)* 0 1 *and* 1 3 *f)* 3 2 *and* 2 1

5. Place the symbols <, =, or > between each pair of integers to obtain true statements:

a) 1 2 *b)* 1 0 *c)* 1 1 *d)* 0 5 *e)* 0 0

f) 3 7 *g)* 4 2 *h)* 2 4 *i)* 5 5 *j)* 7 5

6. Find the logical value (true or false) of each statement:

a) 3 2 *b)* 2 2 *c)* 1 1 *d)* 0 5 *e)* 1 1

f) 5 7 *g)* 4 2 *h)* 3 5 *i)* 1 2 *j)* 3 3

7. Two more comparison operators are (greater or equal operator) and (less or equal operator). Find the logical value (true or false) of each statement:

a) 1 1 *b)* 0 1 *c)* 1 2 *d)* 2 2 *e)* 1 1

f) 2 3 *g)* 5 4 *h)* 3 3 0 *i)* 0 2 *j)* 3 0

8. Find the logical value (true or false) of each statement (Note: a<b<c is true if both a<b and b<c are true):

a) 0 1 2 *b)* 1 0 1 *c)* 1 2 3 *d)* 1 2 0 *e)* 3 1 1

f) 1 1 0 *g)* 5 3 1 *h)* 5 3 1 *i)* 0 1 3 *j)* 3 0 2

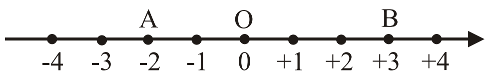
9. Find the set of values of the integer x so each statement is true:

a) 1 *x* 3 *b)* 2 *x* 1 *c)* 1 *x* 3 *d)* 1 *x* 0 *e)* 1 *x* 5

f) 1 *x* 3 *g)* 4 *x* 1 *h)* 0 *x* 4 *i)* 0 *x* 3 *j)* 2 *x* 2

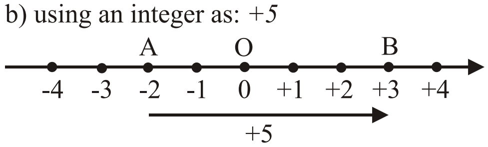
5. Applications of Integers

5.1 *(Position)* Let's suppose that an object is placed at point *A* on the number line.

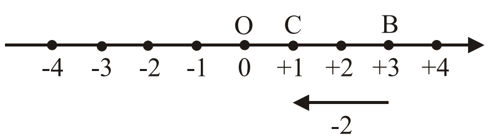


The position of the object can be described:
a) in words as: *2 units left of the origin O*
b) using an integer as: *-2*
If the object is moved from point A to point B, its new position can be described:
a) in words: 3 *units right of the origin O*
b) using an integer: *+3*

5.2 *(Displacement)* A change in position of the object is called *displacement*. The displacement of the object from the point *A* to the point *B* can be described:
a) in words as: *5 units to the right*
b) using an integer as: *+5*



The displacement of the object from point *B* to point *C* can be described:
a) in words: *2 units to the left*
b) using an integer: *-2*



5.3 *(Temperature)* Temperature can be expressed using integers. The temperature of *0°C* corresponds to *0*. A temperature of *20°C* above *0°C* is *+20*. A temperature *30°C* below *0°C* is *-30*. The smallest possible temperature is approximately 2*73°C* below *0°C* and corresponds to *-273*.
An increase in temperature of *10°C* corresponds to *+10* and a decrease in temperature of *40°C* corresponds to *-40*.

5.4 *(Altitude)* Altitude can be expressed using integers. Sea level corresponds to *0*. An altitude of *100m* above the sea level corresponds to *+100*. A depth of *50m* below sea level corresponds to *-50*. Rising *200m* corresponds to *+200* and falling *500m* corresponds to *-500*.

5.5 *(Bank Account)* Bank account can be expressed using integers. A debit of *$100* corresponds to *+100*. A credit of *$25* corresponds to *-25*. A deposit of *$50* corresponds to *+50* and a withdraw of *$75* corresponds to *-75*.

1. Use integers to describe the position of an object. The object is positioned:

A) 5 units right of the origin *B) 4 units left of the origin*
C) in the origin *D) 1 unit right*
E) 2 units left

2. Use integers to describe the displacement of an object. The object is:

A) moved 2 units to the right *B) moved 2 units to the left*
C) fixed in the origin *D) moved 3 units right*
E) moved 5 units left

3. Use integers to describe the outside temperature. The temperature:

A) is 10 degrees above 0 *B) is 15 degrees below 0*
C) for melting ice *D) for boiling water*
E) the lowest possible

4. Use integers to describe the change in the temperature. The temperature is:

A) increasing by 10 degrees *B) decreasing by 5 degrees*
C) the same *D) 3 degrees more*
E) 10 degrees less

5. Use integers to describe the position of an object relative to sea level. The object is:

A) 1000m above *B) 2m below*
C) 20m under sea level *D) at sea level*
E) 7m higher than sea level

6. Use integers to describe the change in the altitude of an object. The object is:

A) raised 2m *B) lowered 5m*
C) moved 7m upward *D) moved 3m downward*
E) not moved at all

7. Use integers to describe the balance of a bank account. The balance is:

A) a debit of $500 *B) a credit of $150*
C) the initial value *D) a debt of $31*
E) $125 cash

8. Use integers to describe the transaction in a bank account. The transaction is:

A) a deposit of $50 cash *B) a deposit of a $550 cheque*
C) a withdraw of $40 cash *D) a payment of a $45 bill*
E) just updating the address

6. Addition of Integers. Axioms

6.1 *(Addition)* Addition is a *binary operation* (represented by the symbol +) between two operands (called *terms* or *addends*) to obtain a result called the *sum*. Example:

(3) (2) 5

The terms (addends) are *3* and *2*.
The sum is *5*.

6.2 *(Commutative Property)* The order in which the addends are added is not important. This is the *commutative property* of addition. This property can be written as:

a b b a

Example:

(2) (4) (4) (2) 6

6.3 *(The Additive Identity) 0* is the *additive identity* of addition satisfying the property:

a 0 0 *a a*

Example:

(5) 0 0 (5) 5

6.4 *(Opposites Cancellation)* The sum of a number (integer) and its opposite is *0*:

a (*a*) (*a*) *a* 0

Example:

(3) (3) 0

6.5 *(The Associative Property)* If an expression contains more than 2 terms then the order in which the additions are done is not important. For 3 terms this property states that:

(*a b*) *c a* (*b c*)

Example:

1 (2 3) 1 5 6

(1 2) 3 3 3 6

6.6 *(Application)* By combining the previous properties the sum of a complex addition (involving more terms) might be calculated very easily.
Example:

(2) (4) (2) (7) (4)

[(2) (2)] [(4) (4)] (7)

0 0 (7) 7

1. For each case, identify the addends and the sum:

a) 1 2 3 *b*) (2) (1) 1 *c*) (2) (2) 4

d) 7 (3) 4 *e*) 4 (6) 10 *f*) 6 2 4

2. Use the commutative property (see 6.2) to rewrite these additions:

a) 0 1 *b*) 2 3 *c*) 5 0

d) (1) 0 *e*) (3) 4 *f*) (4) (5)

3. Use the additive identity property (see 6.3) to find each sum:

a) 0 0 *b*) 0 1 *c*) 5 0

d) (1) 0 *e*) (4) 0 *f*) 0 (5)

4. Use the opposites cancellation property (see 6.4) to find each sum:

a) 1 (1) *b*) (2) (2) *c*) 5 (5)

d) (10) (10) *e*) (0) (0) *f*) (7) (7)

g) (5) 0 5 *h*) 3 0 (3) *i*) 0 (11) 0 (11)

5. Use the associative property (see 6.5) to find each sum easily. One case is solved for you as an example. You might also need to apply the commutative property.

a) (5 3) 7 *b*) (5 (2)) (2) *c*) (5 19) 1

d) (5) (5 7) *e*) 14 (16 3) *f*) 17 ((17) 37)

g) (3) (5) 5 *h*) 3 (7) (7) *i*) 11 5 (11)

Example:

i) 11 5 (11) (11 5) (11) (11) (11 5)
((11) (11)) (5) 0 5 5

6. Group the terms conveniently (see 6.6) to find the sum. One case is solved for you as an example:

a) 1 (2) (1) *b*) (3) 5 3 *c*) (4) (2) (4) (2)

d) 1 2 3 (1) (2) *e*) 4 (10) 6 *f*) 3 (5) 2 5

g) 1 (3) 9 3 (1) *h*) 2 4 6 8 *i*) 2 4 (6) 8

j) 11 (3) 19 *k*) 17 (4) 13 3 1 *l*) 2 (9) 3 (5) 4 5

Example:

l) 2 (9) 3 (5) 4 5 ((2 3) (5)) ((9) (4 5))
(5 5) ((9) 9) 0 0 0

7. Addition of integers using the number line

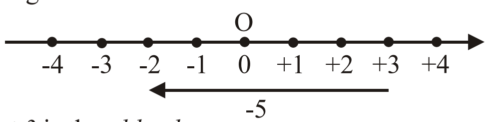
7.1 *(Addition Redefined)* The addition operation can be interpreted as:

old value + change = new value

Example. Let's find the sum of:

(3) (5)

using this new definition of addition together with the number line:



+3 is the *old value*
-5 is the *change*
-2 is the *new value*
So:

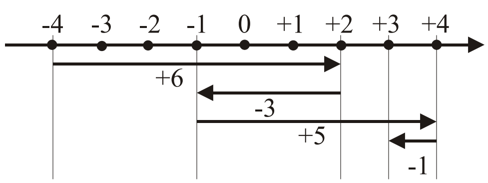
(3) (5) 2

7.2 *(More addends)* If more terms (addends) are to be added, then the meaning of the addition operation is:

old+change+change+...+change=new

Example. Let's find the sum of:

(4) (6) (3) (5) (1)



So:
(4) (6) (3) (5) (1) 3

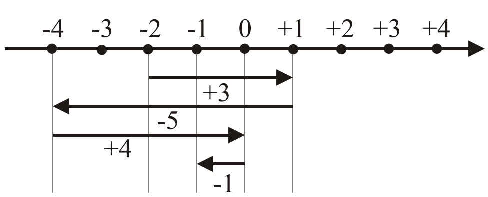
7.3 *(The hidden addition sign)* When adding more numbers, usually the addition operator (+) between the numbers can be dropped without affecting the clarity of the expression. So, an expression as:

2 3 5 4 1

is in fact equivalent of:

(2) (3) (5) (4) (1)

Let's use number line to find the sum:



So: 2 3 5 4 1 1

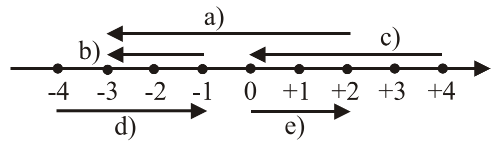
1. Use the number line to find each sum:

a) 0 (2) *b*) 0 (3) *c*) (3) (2) *d*) (1) (2)

e) (4) (3) *f*) (2) (3) *g*) (4) (1) *h*) (2) (1)

i) (3) (5) *j*) (2) (6) *k*) (2) 0 *l*) (3) (5)

2. Find the addition operation that corresponds to each arrow on the number line:



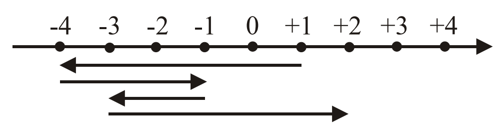
3. Use the number line to find each sum:

a) (1) (2) (3) *b*) (3) (1) (3) *c*) (2) (5) (3)

d) (1) (2) (3) *e*) 0 (1) (2) (3) *f*) (2) (4) (6)

g) (1) (2) (1) *h*) (5) (2) (1) (3) *i*) (2) (2) (3) (3)

4. Find the addition operation that corresponds to the sequence of arrows on the number line:



5. Simplify the following expressions by hiding the addition operators between two consecutive terms. Do not calculate the sum.

a) (1) (2) *b*) (3) (2) *c*) (2) (5)
d) (2) (2) (3) *e*) (5) (3) (1) *f*) (1) (2) (4) (5)

6. Rewrite each expression by revealing the hidden addition operators. Do not calculate the sum.

a) 1 2 *b*) 2 3 *c*) 3 5
d) 1 2 3 4 *e*) 2 2 3 3 *f*) 5 4 3 2 1 2

7. Use the number line to calculate each sum:

a) 1 3 5 *b*) 1 2 3 *c*) 5 4 3 2

d) 2 2 2 *e*) 1 3 5 *f*) 0 2 4 5

g) 2 3 4 5 *h*) 2 4 3 5 *i*) 1 2 3 4 5 6 7

j) 1 2 4 8 *k*) 1 3 6 10 *l*) 1 3 5 2 4 6

8. Find the value of each expression. One case is solved for you as an example:

a) |2 5| *b*) 2 | 3| *c*) | 2| | 5| *d*) 5 | 6| | 2|

Example:

d) 5 | 6| | 2| 5 6 2 1 2 3

8. Addition of integers using rules

8.1 *(Addition Rules)* The method of using the number line to add two or more integers is not useful for adding large numbers (in absolute value). In these cases the addition rules presented bellow must be used.

8.2 *(The Addition of Positive Integers)* If you add two or more *positive integers*:
a) the *sign* of the result is *positive*
b) the *absolute value* of the result is the *sum* of the absolute values of each of the terms

(a) (b) (c) ... (a b c ...)

Examples:

(2) (5) (2 5) 7
(3) (2) (4) (3 2 4) 9

8.3 *(The Addition of Negative Integers)* If you add two or more *negative integers*:
a) the *sign* of the result is *negative*
b) the *absolute value* of the result is the *sum* of the absolute value of each of the terms

(a) (b) (c) .. (a b c ...)

Examples:

(1) (3) (1 3) 4
(3) (1) (4) (3 1 4) 8
1 2 3 4 (1 2 3 4) 10

8.4 *(The Addition of Integers with Opposite Signs)* If you add two integers with opposite signs:
a) the *sign* of the sum is the sign of the integer with the *biggest absolute value*
b) the *absolute value* is the *difference* between the greatest absolute and the smallest absolute value
Case 1. If the absolute value of the positive number is greater than the absolute value of the negative number:

(A) (b) (A b)

Examples:

(5) (3) (5 3) 2
3 6 (6 3) 3

Case 2. If the absolute value of the negative number is greater than the absolute value of the positive number:

(a) (B) (B a)

Examples:

(2) (7) (7 2) 5
4 1 (4 1) 3

1. For each case, calculate the sum of the positive integers (see 8.2):

a) 1 3 *b*) (3) (7) *c*) 5 (17) *d*) 21 33
e) 101 24 *f*) 321 123 *g*) 200 (35) *h*) (25) (375)
i) 1000 135 *j*) 1250 325 *k*) 550 3525 *l*) 12345 54321
m) 1 2 3 *n*) 10 25 30 *o*) 10 200 3000 *p*) 1 22 333

2. For each case, calculate the sum of the negative integers (see 8.3):

a) 1 2 *b*) (2) (3) *c*) (3) (3)
d) 200 (25) *e*) (125) (125) *f*) 450 (350)
g) 1000 3500 *h*) 2222 3333 *i*) 1234 100
j) 2 3 4 *k*) 10 25 55 *l*) 200 50 150 1500 5

3. In each of the following cases, the absolute value of the positive integer is greater than the absolute value of the negative integer. Find the sum:

a) 5 3 *b*) (4) (2) *c*) (3) (6) *d*) 5 7
e) 10 5 *f*) 15 20 *g*) (20) (55) *h*) (55) (15)
i) 100 50 *j*) 25 125 *k*) (100) (300) *l*) (550) (125)
m) 2500 1250 *n*) 250 300 *o*) (1111) (3333) *p*) (150) (50)

4. In each of the following cases, the absolute value of the negative integer is greater than the absolute value of the positive integer. Find the sum:

a) (3) (4) *b*) (7) (3) *c*) 5 8 *d*) 9 4
e) 3 10 *f*) 25 10 *g*) (30) (25) *h*) (15) (40)
i) 50 200 *j*) 150 125 *k*) (600) (150) *l*) (150) (250)
m) 150 2000 *n*) 5500 400 *o*) (4400) (2200) *p*) (125) (350)

5. For each case, calculate the sum:

a) 1 2 3 4 *b*) 0 2 4 *c*) 5 5 6
d) 10 20 50 *e*) 1 10 100 *f*) 100 1000 100
g) 200 20 2 *h*) 5 0 15 *i*) (125) (125) (300)
j) 125 325 0 *k*) 0 5 15 *l*) (1) (3) (5) (1) (3) (2)

6. Group the terms conveniently to find more easily each sum. One case is solved for you as an example:

a) 3 4 7 16 *b*) 3 8 2 7 *c*) 5 4 5 6 4
d) 5 10 3 2 15 *e*) 34 25 66 15 *f*) 5 10 15 20 25

Example:

e) 34 25 66 15 (34 66) (25 15) 100 10 (100 10) 90

9. Subtraction of integers

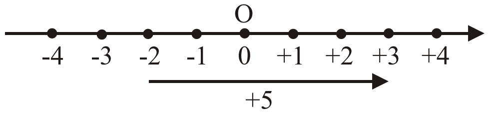
9.1 *(Subtraction)* Subtraction is a *binary operation* between an operand called the *minuend* and an operand called the *subtrahend* to get a result called the *difference*:

minuend subtrahend difference

9.2 *(Number Line)* You can use the number line to calculate the difference between two integers. To do that use this rule:

new value old value change

Example: (3) (2)



where:
-2 is the old value
+3 is the new value
+5 is the change from the old value to the new value
So:

(3) (2) 5

9.3 *(Subtraction Redefined)* Subtraction between a *minuend* and a *subtrahend* can be redefined as an *addition* between the minuend and the *opposite* of the subtrahend:

a b a (*b*)

Example:

(4) (5) (4) (5) 9

9.4 *(Anti-commutative Property)* The subtraction operation has the anti-comutative property:

a b (*b a*)

Example:

(3) (5) (3) (5) 2
(5) (3) 2

9.5 *(Sign Rules)* The following *sign rules* can be useful when dealing with addition and subtraction of integers:

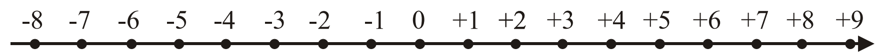
(*a*) *a a*
(*a*) *a*
(*a*) *a*
(*a*) *a a*

Examples:

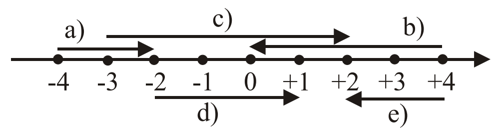
((3)) (3) 3
(5) (4) 5 4 9
(3) (5) 3 5 5 3 2

1. Use the number line to find each difference (see 9.2):

a) (2) (1) *b*) (3) (1) *c*) (3) (2) *d*) (2) (2)
e) (5) (1) *f*) (2) (3) *g*) (4) (1) *h*) 0 (2)
i) (3) (5) *j*) 2 5 *k*) 2 (5) *l*) (7) (5)



2. Find the subtraction operation that corresponds to each arrow on the number line (see 9.2):



3. Rewrite each subtraction as an addition with the opposite, then find the result:

a) 1 5 *b*) 0 4 *c*) 5 3 *d*) 5 (2)
e) (3) (2) *f*) (2) (4) *g*) (4) (1) *h*) 3 (1)
i) (3) (5) *j*) (2) (3) *k*) (7) (2) *l*) 5 (7)

4. Use the anti-commutative property to calculate the differences (see 9.3):

a) 3 7 *b*) 0 4 *c*) (3) (8) *d*) 4 14
e) 7 10 *f*) (6) (12) *g*) 0 (5) *h*) 123 143

5. Use the sign rules to simplify the following expressions (see 9.4):

a) (5) *b*) (3) *c*) (3) *d*) ((2))
e) (4) *f*) ((6)) *g*) ((5)) *h*) (((5)))

6. Find the value of each expression. One case is solved for you as an example:

a) (5) (3) *b*) (3) ((2)) *c*) (3) ((5))
d) ((4)) ((6)) *e*) ((6)) (((4))) *f*) ((2)) ((6))
g) ((4)) ((1)) *h*) ((5)) ((3)) *i*) ((1)) ((1))

Example:

e) ((6)) (((4))) (6) ((4) 6 4 10

7. Find the value of each expression. One case is solved for you as an example:

a) | 2| | 5| *b*) | 4| | 3| *c*) | | 5| 7| *d*) | 1| | 2| | 3|

Example:

c) | | 5| 7| | 5 7| | 2| 2

10. Order of operations (I)

10.1 *(Priority)* The addition and the subtraction are considered operations of the *same priority*. Therefore, if more operations appear within an expression then the operations must be done in *the order they occur* (from *left to right*). Example:

2 5 1 4 3

3 1 4 3

4 4 3 0 3 3

10.2 *(Grouping)* Sometimes it is useful to group all the positive terms and negative terms separately:

a b c d e (*a c e*) (*b d*)

Example:

6 3 4 2 5 7

(6 4 2) (3 5 7)

12 15 5

10.3 *(Brackets)* To change the order of operations, you can use *brackets*. If brackets appear within an expression then the operations inside of the brackets must be done first. Example:

1 (2 5) (3 4)

1 (3) 1 1 3 1 5

10.4 *(Nested Brackets)* In complex expressions the brackets can be *nested* one inside of another. In this case start with the *inner-most* operations . This rule might be applied more than once if necessary. Example:

2 (2 (1 2) 3) (2 3 (4 6))

2 (2 (1)) (2 3 (2))

2 (2 1) (2 3 2)

2 1 (1) 4

10.5 *(Standard Grouping Symbols)* To differentiate nested grouping symbols, three types of *standard grouping symbols* are used: ( ) called *parentheses*, [ ] called square *brackets*, and { } called *braces*. By convention the order of nesting is:

{ [( )] }

Example:

2 {5 [3 (2 3) (5 6 1) 1] 2}

2 {5 [3 (1) (0) 1] 2}

2 {5 [5] 2} 2 {12} 10

1. Use the "left to right" rule (see 10.1) to find the result of each sequence of operations:

a) 1 2 3 *b*) 3 4 1 *c*) 2 3 4

d) 1 2 3 4 *e*) 2 3 4 5 *f*) 5 4 3 2

g) 5 6 7 8 2 *h*) 2 3 4 5 6 *i*) 5 10 15 20 5

2. Use grouping of positive and negative terms (see 10.2) to find the result of each sequence of operations:

a) 3 2 5 *b*) 2 4 1 *c*) 2 3 4

d) 3 4 1 4 *e*) 1 5 2 3 *f*) 5 4 1 3

g) 1 2 3 4 5 6 *h*) 10 8 6 4 2 *i*) 15 20 25 30 50

3. For each case, first do the operations inside of the brackets, then find the result (see 10.3):

a) 1 (2 4) *b*) 2 (3 5) *c*) 0 (2 5)

d) (2 4) (2 4) *e*) (2 5) (3 4) *f*) 1 (3 5) (4 7)

g) (1 3) (5 2) (2 5) *h*) 0 (2 1) (5 7) *i*) 1 (2 3) (4 5) 2

4. For each case, complete the operations starting with the inner-most brackets (see

a) 2 (3 (1 2)) *b*) (2 (3 5) 1) 2

c) (5 2) ((3 5) 4) *d*) (((3) 1) 1)

e) 2 (3 (5 8)) *f*) (((4 1))) (((4 1)))

g) (3 4) ((5 7) (4 2) 1) *h*) 1 (2 (3 (4 (5 6) 1) 2) 3) 4

5. For each case, find the result using the order of operation for "standard grouping symbols" (see 10.5):

a) 1 [2 (4 7) 1] *b*) (5 7) [3 (9 2) 4]

c) [3 (3 4)] [3 (3 4)] *d*) 2 { (5 7) [5 (6 2)]}

e) {[1 (1 2)] (2 3)} [3 (1 2)] *f*) 1 {2 [3 (2 3)]}

g) 1 { [(1 2) (3 5)] [(5 9) (3 1)]} *h*) 2 { 3 [2 (7 9) 1] 2} 1

6. Find the value of each expression. One case is solved for you as an example:

a) 2 (3 | 5|) *b*) (5 | 7|) (2 | 3|) *c*) | 2| (| 2| | 3|)

d) |3 (4 7)| *e*) (3 7) |3 7| *f*) |3 (4 5) 4|

Example:

c) | 2| (| 2| | 3|) 2 (2 3) 2 (1) 2 1 1

11. Equalities and Inequalities

11.1 *(True/False Statements)* In mathematics ideas are expressed as *statements*. A statement can only be *true* or *false*.
Example: "*3<4*" is a *true* statement.
Example: "*3 is a divisor of 10*" is a *false* statement.

11.2 *(Relational Operators)* Some of the most common types of statements are *binary relations* involving the following *relational operators*:

$a \quad b \qquad a \quad b \qquad a \quad b$

$a \quad b \qquad a \quad b \qquad a \quad b$

Examples:

$3 \quad 0$ *true*; $\quad 0 \quad 2$ *false*

$1 \quad 1$ *true*; $\quad 2 \quad 2$ *true*

11.3 *(Equality)* If the relational operator is "=" then the binary relation is called *equality*. Examples:

$0 \quad 0$ *true*

$2 \quad 3 \quad 5 \quad 4$ *false*

11.4 *(Symmetry)* Equality has the symmetry property that states:

$a \quad b$ *if and only if* $b \quad a$

or: $a \quad b \qquad b \quad a$

"a=b" is true if and only if "b=a" is true.
"a=b" is false if and only if "b=a" is false.
Example:

$2 \quad 5 \quad 3 \qquad 3 \quad 2 \quad 5$

11.5 *(Inequality)* If the relational operator is not "=" then the binary relation is called *inequality*. Examples of inequalities:

$1 \quad 2 \quad 3$ *false*; $\quad 1 \quad 2 \quad 3$ *true*

$1 \quad 2 \quad 2$ *true*; $\quad 5 \quad 7 \quad 3$ *true*

11.6 *(Mirroring)* For any inequality there is an equivalent inequality obtained by *mirroring* of the original:

$a \quad b \qquad b \quad a; \qquad a \quad b \qquad b \quad a$

$a \quad b \qquad b \quad a; \qquad a \quad b \qquad b \quad a$

$a \quad b \qquad b \quad a$

The original inequality is true if and only if the mirrored inequality is true. The original inequality is false if and only if the mirrored inequality is false. Examples:

$1 \quad 0 \qquad 0 \quad 1$

$3 \quad 5 \quad 3 \qquad 3 \quad 3 \quad 5$

$2 \quad 2 \qquad 2 \quad 2$

1. Find the logical value (true or false) of each statement (see 11.1):

a) $1 \quad 2$ *b*) $0 \quad 1$ *c*) 15 *is divisible by* 2

d) 3 *is a divisor of* 9 *e*) $\sqrt{2}$ *is an integer* *f*) $1 \quad 2 \quad 1$

2. Find the logical value (true or false) of each statement containing operational operators (see 11.2):

a) $1 \quad 1$ *b*) $1 \quad 1$ *c*) $2 \quad 1$ *d*) $0 \quad 1$

e) $2 \quad 2$ *f*) $0 \quad 1$ *g*) $0 \quad 3$ *h*) $1 \quad 1$

i) $2 \quad 2$ *j*) $2 \quad 5$ *k*) $15 \quad 25$ *l*) $2 \quad 2$

3. Do the required operations on both sides and find the logical value (true or false) of each of the following equalities (see 11.3):

a) $1 \quad 2 \quad 3$ *b*) $1 \quad 2 \quad 1$ *c*) $2 \quad 4 \quad 5$

d) $3 \quad 1 \quad 2$ *e*) $2 \quad 6 \quad 8$ *f*) $2 \quad 5 \quad 6 \quad 3$

g) $1 \quad 2 \quad 3 \quad 3 \quad 2 \quad 1$ *h*) $2 \quad 3 \quad 5 \quad 3 \quad 4 \quad 1$ *i*) $5 \quad 15 \quad 0 \quad 5$

4. Use the symmetry property (see 11.4) to rewrite each equality and then find the value (true or false) of each statement:

a) $1 \quad 3 \quad 2$ *b*) $2 \quad 4 \quad 7$ *c*) $3 \quad 5 \quad 7$

d) $3 \quad 5 \quad 4 \quad 2$ *e*) $1 \quad 3 \quad 5 \quad 7 \quad 1 \quad 3$ *f*) $0 \quad 3 \quad (4 \quad 7)$

5. Do the required operations on both sides and find the logical value (true or false) of each of the following inequalities (see 11.5):

a) $2 \quad 6 \quad 1 \quad 5$ *b*) $1 \quad 5 \quad 2 \quad 2$

c) $1 \quad (3 \quad 7) \quad 2 \quad (1 \quad 4)$ *d*) $2 \quad 4 \quad 6 \quad 8 \quad 2 \quad 2$

e) $2 \quad (3 \quad 2) \quad 1 \quad 3 \quad (3 \quad 8)$ *f*) $10 \quad (15 \quad 20) \quad 15 \quad (10 \quad 20)$

g) $3 \quad 5 \quad 1 \quad (2 \quad 6)$ *h*) $1 \quad (4 \quad 7) \quad (2 \quad 5) \quad 0$

6. Use mirroring (see 11.6) to rewrite the following inequalities and then find the value (true or false) of each statement:

a) $0 \quad 1$ *b*) $1 \quad 2$ *c*) $2 \quad 2$

d) $2 \quad 3 \quad 5 \quad 4 \quad 6 \quad 3$ *e*) $1 \quad (5 \quad 8) \quad (2 \quad 2)$ *f*) $1 \quad 2 \quad 3 \quad 2 \quad 3 \quad 4$

7. Find the logical value (true or false) of each statement containing operational operators and absolute value function. One case is solved for you as an example:

a) $|\ 3|\ |\ 3|$ *b*) $|\ 4|\ |\ 3|$ *c*) $|\ 2|\ |\ 1|$ *d*) $0 \ |\ 1|$

e) $5 \ |\ 5|$ *f*) $0 \ |\ 1|$ *g*) $0 \ |\ 3|$ *h*) $|\ 1|\ |\ 1|$

Example:

c) $|\ 2|\ |\ 1| \quad 2 \quad 1$ (*false*)

12. Equivalent Equalities and Inequalities

12.1 *(Adding or Subtracting)* By adding (or subtracting) the same number to (from) the left side and to (from) the right side of an equality (inequality) you get an *equivalent* equality (inequality):

a b a c b c

a b a c b c

etc.

Examples:

2 4 2 2 4 4 2 4

(*both are false*)

1 3 2 4 1 3 2 2 4 2

(*both are true*)

12.2 *(Moving Terms)* Moving a term from one side of an equality (inequality) to the other side requires changing the term to its *opposite* in order to get an *equivalent* equality (inequality):

a b c a c b

a b c a c b

Examples (true statements):

1 2 3 2 1 3 2 2

3 1 5 2 3 5 2 1

12.3 *(Empty side)* After removing the last term from one side of an equality (inequality) to the other side *replace the empty side with 0*. Indeed:

a b 0 *a b* 0 *b a*

Examples (true statements):

1 2 3 2 0 2 1 2 3

0 1 2 3 1 2 3 0

12.4 *(Positive Terms)* By moving terms from one side of an equality (inequality) to the other side, it is always possible to have an equivalent equality (inequality) with positive terms on both sides. Example:

2 4 5 2 4 1 4 4 1 2 2 5

12.5 *(Simplifying)* If both sides of an equality (inequality) contain the same term then this term can be *cancelled out*:

a c b c a b

a c b c a b

etc.

Examples (true statements):

5 6 7 5 6 7

1 2 3 4 3 1 2 4

1. Do the required operation, then simplify, and finally find the logical value (true or false) of each statement:

a) 2 2; *add* 2 *b*) 2 3 1; *add* 3 *c*) 2 4 0; *add* 4

d) 7 1 2; *add* 1 *e*) 2 3 5; *add* 5 *f*) 3 6 2; *add* 6

2. By moving terms from one side to the other side (see 12.2), find 3 equivalent equalities (inequalities):

a) 2 3 1 *b*) 2 3 4 *c*) 4 6 1

d) 5 3 2 1 *e*) 2 1 3 3 2 *f*) 10 15 5 15 10

3. Find an equivalent equality (inequality) by moving all terms from the right side to the left side:

a) 2 2 4 *b*) 3 1 4 *c*) 0 1 2

d) 3 2 2 *e*) 1 2 2 3 *f*) 10 15 5 10 15

4. Find an equivalent equality (inequality) by moving all terms from the left side to the right side:

a) 4 7 3 *b*) 1 3 2 *c*) 1 2 1 2

d) 1 1 3 *e*) 5 10 10 15 *f*) 3 7 5 3 2

5. Find an equivalent equality (inequality) by moving terms from one side to the other side so finally each side to have only positive terms (see 12.4):

a) 1 2 1 *b*) 2 1 *c*) 3 2

d) 3 4 *e*) 1 2 4 2 5 *f*) 2 3 4 1 2 3

6. Find an equivalent equality (inequality) by canceling out the identical terms on both sides (see 12.5):

a) 1 2 1 1 1 *b*) 3 3 1 *c*) 2 3 2 2

d) 5 5 1 *e*) 1 2 4 4 1 1 *f*) 2 1 3 3 2 3 5

7. Find an equivalent equality (inequality) so one side is 0 and the other side is an integer. One case is solved for you as an example:

a) 3 1 2 *b*) 2 4 1 2 *c*) 5 3 1 2

d) 1 (2 3) (3 2) *e*) (2 (4 8) 1) 2 6 *f*) (3 5) (5 8) 2

Example:

e) (2 (4 8) 1) 2 6 7 8 7 8 0 1 0 (*true*)

8. Find the logical value (true or false) of each statement. One case is solved for you as an example:

a) | 2| | 2| *b*) |0| | 3| *c*) | 5| | 2| *d*) 1 | 1| 0

e) | 7| | 7| *f*) | 8| | 8| *g*) | 1| | 3| *h*) | 1| | 2| 1 2

Example:

h) | 1| | 2| 1 2 1 2 3 3 3 (*true*)

13. Equations

13.1 *(Equation)* An *equation* is an equality containing an *unknown number* represented by a letter and called *variable*. Example:

$$1 \quad x \quad 2 \qquad 5$$

In this equation x is the unknown number or variable.

13.2 *(Solution)* A *solution* of an equation is the *value* (number) of the variable that makes the equation a *true statement*. If an equation has more solutions, these solutions together form a *set solution*. Example:

$$x \quad 1 \qquad 3$$

A solution of this equation is $x = -4$. Indeed, if you replace x by *-4* in the equation, you get a true statement:

$$4 \quad 1 \qquad 3$$

13.3 *(Solving)* To solve an equation means to write a *sequence of equivalent equations* until you *isolate* the variable. Example:

$$2 \quad x \quad 3 \quad 4 \quad 6$$

a) regroup the left side (LS) and simplify the right side (RS): $x \quad (2 \quad 3) \qquad 2$

b) simplify LS: $x \quad 1 \qquad 2$

c) move *-1* from LS to RS: $x \qquad 2 \quad 1$

d) simplify RS: $x \qquad 1$

13.4 *(Variable on RS)* If the variable appears on the RS of an equation, then use the symmetry property of an equality. Example:

$$2 \quad 5 \quad 1 \quad x \quad 3 \qquad 3 \quad x \quad 2$$
$$3 \quad 2 \quad x \qquad 5 \quad x \qquad x \quad 5$$

13.5 *(Opposite Variable)* If *-x* appears instead of x in the equation, then move *-x* to the other side. Example:

$$1 \quad x \quad 2 \qquad 1 \quad 2 \quad x \qquad 1 \quad 2 \quad x$$
$$1 \quad x \qquad x \qquad 1$$

13.6 *(Brackets)* If the equation contains brackets, then *simplify and expand (remove) the brackets* until you succeed in isolating the variable. Example:

$$2 \quad (3 \quad (2 \quad x)) \quad 5$$
$$\textit{expand:} \quad 2 \quad (3 \quad 2 \quad x) \quad 5$$
$$\textit{simplify:} \quad 2 \quad (1 \quad x) \quad 5$$
$$\textit{expand:} \quad 2 \quad 1 \quad x \quad 5$$
$$\textit{simplify:} \quad 1 \quad x \quad 5$$
$$\textit{move x to RS:} \quad 1 \quad 5 \quad x$$
$$\textit{simplify:} \quad 4 \quad x \quad \textit{mirror:} \quad x \quad 4$$

1. Find if the given value of x is a solution of the following equations (use substitution as explained in 13.2).

$a) \quad x \quad 0; x \quad 0 \qquad b) \quad x \quad 1 \quad 2; x \quad 1 \qquad c) \quad x \quad 2 \quad 3; x \quad 0$

$d) \quad 2 \quad 3 \quad x \quad 1; x \quad 2 \qquad e) \quad 1 \quad x \quad 1 \quad x; x \quad 2 \qquad f) \quad 1 \quad (2 \quad x) \quad 5; x \quad 4$

2. Solve for x by isolating the variable on the left side and moving all the other terms to the right side (see 13.3):

$a) \quad x \quad 1 \quad 2 \qquad b) \quad x \quad 3 \quad 1 \qquad c) \quad 1 \quad x \quad 5$

$d) \quad 3 \quad x \quad 5 \qquad e) \quad 2 \quad x \quad 3 \quad 1 \quad 3 \qquad f) \quad 10 \quad x \quad 20 \quad 30 \quad 40$

$g) \quad 2 \quad 1 \quad x \qquad h) \quad 3 \quad 5 \quad 1 \quad x \quad 2 \qquad i) \quad 0 \quad x \quad 3 \quad 5$

3. Solve for x by isolating the variable on the right side and moving all the other terms to the left side (see 13.4):

$a) \quad 1 \quad x \quad 2 \qquad b) \quad 2 \quad 3 \quad x \qquad c) \quad 5 \quad x \quad 5$

$d) \quad 1 \quad 1 \quad x \quad 2 \qquad e) \quad 2 \quad 2 \quad 3 \quad x \qquad f) \quad 15 \quad 10 \quad x \quad 20$

$g) \quad 2 \quad x \quad 3 \qquad h) \quad x \quad 1 \quad 2 \quad 3 \qquad i) \quad 5 \quad x \quad 10 \quad 15$

4. Use the "simplify and expand" method explained in 13.6 to solve the following equations for x:

$a) \quad 1 \quad (x \quad 1) \quad 2 \qquad b) \quad 2 \quad (2 \quad x) \quad 1$

$c) \quad 3 \quad 5 \quad (x \quad 2) \qquad d) \quad (3 \quad 2) \quad (3 \quad x) \quad 0$

$e) \quad 2 \quad [4 \quad (3 \quad x)] \quad 1 \quad (2 \quad 3) \qquad f) \quad 5 \quad 10 \quad [5 \quad (\quad x \quad 15)]$

5. Solve for x using the "working backward" method (the first case is solved for you as an example):

$a) \quad 2 \quad (1 \quad x) \quad 3 \qquad b) \quad 10 \quad 5 \quad (15 \quad x)$

$c) \quad (2 \quad 5) \quad (x \quad 5) \quad 1 \quad 0 \qquad d) \quad 1 \quad (2 \quad (x \quad 5) \quad 1) \quad 1$

$e) \quad 1 \quad (2 \quad (3 \quad x) \quad 1) \quad 2 \quad 5 \qquad f) \quad 1 \quad (4 \quad 7) \quad 2 \quad (3 \quad 5 \quad (4 \quad (\quad x) \quad 1))$

Example:

$a) \quad 2 \quad (1 \quad x) \quad 3 \qquad (1 \quad x) \quad 3 \quad 2 \qquad (1 \quad x) \quad 5 \qquad 5 \quad 1 \quad x \qquad x \quad 1 \quad 5 \qquad x \quad 4$

6. Solve for x. One case is solved for you as an example:

$a) \quad |x| \quad 1 \qquad b) \quad |x| \quad 0 \qquad c) \quad |x| \quad 2 \quad 0$

$d) \quad | \quad x| \quad 5 \qquad e) \quad 1 \quad |x| \quad 6 \qquad f) \quad |x| \quad | \quad 5|$

$g) \quad |x| \quad 2 \quad 1 \qquad h) \quad |x| \quad | \quad 1| \quad | \quad 2| \qquad i) \quad |x| \quad | \quad 1| \quad | \quad 2| \quad | \quad 3|$

Example:

$i) \quad |x| \quad | \quad 1| \quad | \quad 2| \quad | \quad 3| \qquad |x| \quad 1 \quad 2 \quad 3 \qquad |x| \quad 3 \quad 1 \quad 2 \qquad |x| \quad 2 \qquad x \quad 2$

14. Inequations

14.1 *(Inequation)* An *inequation* is an inequality containing an *unknown number* represented by a letter and called *variable*. Example:

$$x \quad 2 \quad 5$$

14.2 *(Solution)* A specific *value* of the variable that makes the inequation a *true statement* is called a *solution* of the inequation. Example:

$$1 \quad x \quad 5$$

x = 1 is a solution of this inequation. Indeed, by replacing *x* with *1*, the previous inequation become a true statement:

$$1 \quad 1 \quad 5 \qquad 0 \quad 5$$

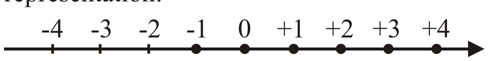
14.3 *(Set Solution)* In general, an inequation has more than one solutions. The *set solution* is the set of all solutions of a given inequation. Example:

$$x \quad 1; \quad x \quad Z$$

The set solution is:

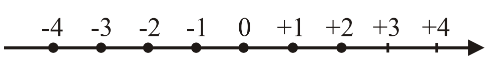
$$x \quad \{ \ 1,0, \ 1, \ 2, \ 3, \ 4,...\}$$

and has the following graphical representation:



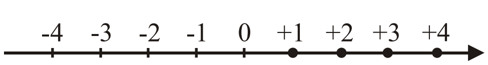
14.4 *(Solving)* Solving an inequation implies finding a *sequence of equivalent inequations* until the variable is isolated. Example:

$$2 \quad x \quad 3 \quad 1; \quad x \quad Z$$
$$3 \quad 2 \quad 1 \quad x$$
$$2 \quad x$$
$$x \quad 2$$



14.5 *(Brackets)* If an inequation contains brackets, then *simplify and expand (remove) the brackets* until the variable is isolated. Example:

$$5 \quad [3 \quad (2 \quad x)] \quad 4$$
$$5 \quad [3 \quad 2 \quad x] \quad 4$$
$$5 \quad [1 \quad x] \quad 4$$
$$5 \quad 1 \quad x \quad 4$$
$$5 \quad 1 \quad 4 \quad x$$
$$0 \quad x$$
$$x \quad 0$$



1. Find if the given value of x is a solution of the given inequation (use substitution as explained in 14.2).

$a) \quad x \quad 1 \quad 0; x \quad 2 \qquad b) \quad x \quad 2 \quad 1; x \quad 2 \qquad c) \quad 3 \quad x \quad 2; x \quad 2$

$d) \quad x \quad 3 \quad 0; x \quad 4 \qquad e) \quad 5 \quad 2 \quad x; x \quad 3 \qquad f) \quad 2 \quad 4 \quad (x \quad 2); x \quad 1$

2. On a number line, graphically represent the set solution of each inequality:

$a) \quad x \quad 2 \quad b) \quad x \quad 3 \quad c) \quad x \quad 1 \quad d) \quad x \quad 1 \text{ and } x \quad 3 \quad e) \quad 3 \quad x \quad 3$

$f) \quad x \quad 2 \quad g) \quad x \quad 1 \quad h) \quad x \quad 0 \quad i) \quad x \quad 2 \text{ and } x \quad 3 \quad j) \quad x \quad 2 \text{ and } x \quad 1$

3. Solve for x by isolating the variable on the left side:

$a) \quad x \quad 2 \quad 0 \qquad b) \quad 2 \quad x \quad 1 \qquad c) \quad x \quad 3 \quad 2$

$d) \quad 1 \quad x \quad 3 \quad 2 \quad 5 \qquad e) \quad x \quad 1 \quad 2 \quad 3 \quad 5 \qquad f) \quad 30 \quad x \quad 20 \quad 10 \quad 15$

4. Solve for x by mirroring the inequation:

$a) \quad 1 \quad x \quad b) \quad 1 \quad x \quad c) \quad 5 \quad x \quad d) \quad 0 \quad x$

$e) \quad 7 \quad x \quad f) \quad 5 \quad x \quad g) \quad 0 \quad x \quad h) \quad 3 \quad x$

5. Solve for x by isolating the variable on the right side, simplifying, and then mirroring the inequation (the first case is solved for you as an example):

$a) \quad 0 \quad x \quad 4 \qquad b) \quad 3 \quad x \quad 5 \qquad c) \quad 1 \quad 2 \quad 3 \quad x$

$d) \quad 4 \quad 7 \quad 2 \quad x \quad 3 \qquad e) \quad 5 \quad 8 \quad x \quad 3 \quad 5 \qquad f) \quad 5 \quad 10 \quad 15 \quad 5 \quad x \quad 10$

Example: $a) \quad 0 \quad x \quad 4 \quad 0 \quad 4 \quad x \quad 4 \quad x \quad x \quad 4$

6. Solve for x by moving the variable to the other side, simplifying and eventually mirroring the inequality:

$a) \quad 0 \quad x \quad 3 \qquad b) \quad x \quad 5 \quad 1 \quad 4 \qquad c) \quad 1 \quad x \quad 3 \quad 5$

$d) \quad 1 \quad 5 \quad 3 \quad x \quad 1 \qquad e) \quad x \quad 9 \quad 1 \quad 2 \quad 4 \qquad f) \quad 5 \quad 10 \quad 10 \quad x \quad 15$

7. Solve for x using the "expand the brackets and simplify" method (see 14.5):

$a) \quad 3 \quad (x \quad 2) \quad 5 \qquad b) \quad 5 \quad (1 \quad x) \quad 3$

$c) \quad 5 \quad 5 \quad (x \quad 5) \qquad d) \quad (\ 3 \quad 2) \quad (\ 5 \quad x) \quad 2$

$e) \quad 1 \quad [2 \quad (\ 1 \quad x)] \quad 3 \quad (\ 5 \quad 2) \qquad f) \quad 5 \quad 15 \quad [10 \quad (\ x \quad 25)]$

8. Solve for x using the "working backward" method (the first case is solved for you as an example):

$a) \quad 3 \quad 2 \quad (x \quad 3) \qquad b) \quad 5 \quad (3 \quad x) \quad 2 \quad 7$

$c) \quad 1 \quad 4 \quad (2 \quad x \quad 5) \qquad d) \quad (\ x \quad 2) \quad 3 \quad 2 \quad 1$

$e) \quad (5 \quad 7) \quad [\ (x \quad 3) \quad 2] \quad 0 \qquad f) \quad [1 \quad (2 \quad 5)] \quad [3 \quad (x \quad 1)] \quad [\ (3 \quad 4) \quad 1]$

Example:

$a) \quad 3 \quad 2 \quad (x \quad 3) \quad 3 \quad 2 \quad (x \quad 3) \quad 5 \quad (x \quad 3) \quad x \quad 3 \quad 5$
$x \quad 5 \quad 3 \quad x \quad 8$

15. Multiplication of Integers (I)

15.1 *(Shortcut)* Multiplication was initially designed as a shortcut for addition of like terms:

$$\underbrace{a \quad a \quad a \quad ... \quad a}_{n\ times} \quad n \quad a$$

n is called the *multiplier*, *a* is called the *multiplicand*, and the result of the multiplication operation is called the *product*. Example:

$$\underbrace{2 \quad 2 \quad 2 \quad 2 \quad 2}_{5\ times} \quad 5 \quad 2 \quad 10$$

15.2 *(Commutativity)* The multiplication operation is *commutative* (the multiplier and the multiplicand can be interchanged without affecting the product):

$$a \quad b \quad b \quad a$$

Example:

$$5 \quad 3 \quad \underbrace{3 \quad 3 \quad 3 \quad 3 \quad 3}_{5\ times} \quad 15$$

$$3 \quad 5 \quad \underbrace{5 \quad 5 \quad 5}_{3\ times} \quad 15$$

15.3 *(Multiplying positive)* The product of two positive integers is positive:

$$(\ a) \ (\ b) \quad (\ b) \ (\ a) \quad (a \quad b)$$

15.4 *(Multiplying integers with opposite signs)* The product of two integers with opposite signs is negative:

$$a \ (\ b) \quad (\ b) \ a \quad (a \quad b)$$

Example:

(3) (3) 6

(3) (3) 2 (3)

So: 2 (3) 6 (2 3)

15.5 *(Multiplying negative integers)* The product of two negative integers is positive:

$$(\ a) \ (\ b) \quad (\ b) \ (\ a) \quad a \quad b$$

Example: (3) (4) (3 4) 12

15.6 *(Multiplying by 0)* The product of any number and *0* is *0*:

$$a \quad 0 \quad 0 \quad a \quad 0$$

Example: (7) 0 0 (7) 0

15.7 *(Multiplicative Identity)* The product between a number and 1 is equal to that number (1 is called the *multiplicative identity*):

$$a \quad 1 \quad 1 \quad a \quad a$$

Example: (3) 1 1 (3) 3

1. Use the short notation for the following additions:

a) 1 1 1 *b*) 5 5 5 *c*) 3 3 3 3 3

d) (2) (2) (2) *e*) 1 1 1 1 1 1 *f*) 7 7 7 7 7 7 7

2. Use the expanded notation for the following multiplications:

a) 3 2 *b*) 4 (5) *c*) 2 (1) *d*) 5 (5) *e*) 3 0

f) 5 (3) *g*) 3 (4) *h*) 5 (10) *i*) 1 (1) *j*) 3 (18)

3. Use the commutativity property of multiplication to expand the following multiplications in two different ways (see 15.2):

a) 1 5 *b*) 2 3 *c*) 4 2 *d*) 6 2 *e*) 5 3

f) 5 2 *g*) 3 4 *h*) 3 3 *i*) 1 1 *j*) 0 2

4. Multiply the following positive integers (see 15.3):

a) 1 1 *b*) 2 (5) *c*) (3) 5 *d*) 3 (10)

e) (3) (2) *f*) 10 10 *g*) (5) (10) *h*) (6) (5)

5. Multiply the following integers with opposite signs (see 15.4):

a) 2 (3) *b*) (1) (5) *c*) (3) 1 *d*) (2) (2) *e*) 6 (2)

f) 1 (1) *g*) 3 (3) *h*) (4) 3 *i*) (10) 2 *j*) (5) (6)

6. Multiply the following negative integers (see 15.5):

a) 1 (1) *b*) (2) (3) *c*) (3) (1) *d*) (5) (2) *e*) (2) (4)

f) 5 (4) *g*) (3) (4) *h*) (1) (9) *i*) (10) (2) *j*) (5) (5)

7. Multiply the following integers (see 15.6 and 15.7):

a) 0 (1) *b*) 0 (5) *c*) 0 0 *d*) 1 1

e) 1 (1) *f*) 10 1 *g*) 1 0 *h*) 1 (10)

8. Multiply the following integers:

a) 5 4 *b*) 5 (4) *c*) (5) 4 *d*) (5) (4)

e) 10 10 *f*) 10 (10) *g*) (10) 10 *h*) (10) (10)

i) (5) (20) *j*) (30) (2) *k*) (5) (40) *l*) (5) (25)

m) 6 6 *n*) (6) (20) *o*) (6) (50) *p*) (6) (50)

q) (123) (1) *r*) (531) (1) *s*) 0 (2468) *t*) (111) 1

9. Find the value of each expression:

a) |2 (3)| *b*) |(2) (4)| *c*) |0 (5)| *d*) |(333) (1)|

e) 1 | 2| *f*) | 3| | 3| *g*) | 5| 0 *h*) (5432) |0|

16. Multiplication of Integers (II)

16.1 *(Divisors)* The operation of multiplication has a special meaning when the multiplicand and the multiplier are integers:

$a \quad b \quad c$

a and *b* are called *factors* or *divisors* of the product *c*.
c is a *multiple* of *a* and *b*.
Example:

(3) (5) 15

-3 and *+5* are factors or divisors of *-15*.
-15 is a multiple of *-3* and *+5*.

16.2 *(Associativity)* The operation of multiplication has the property of associativity (the order in which 3 numbers are multiplied is not important):

$a \quad b \quad c \quad a \quad (b \quad c) \quad (a \quad b) \quad c$

By convention, if more than one multiplication operations appear, the operations are done in the order they occur, from left to right. Example:

(3) (2) (4) (6) (4) 24

16.3 *(Distributivity)* The multiplication operation is *distributive* over the addition operation:

$a \quad (b \quad c) \quad a \quad b \quad a \quad c$

Example. The expression:

(3) [(5) (4)]

can be evaluated in two different ways:

(3) [1] 3

(3) (5) (3) (4) 15 12 3

16.4 *(Positive Factor)* If the number in front of the bracket is positive, the signs of the final terms are *coincident* with the signs of the terms inside of the brackets:

$a \quad (b \quad c \quad d \quad e) \quad a \quad b \quad a \quad c \quad a \quad d \quad a \quad e$

Example:

2 (3 4 5)

2 (4) 8

2 3 2 4 2 5 6 8 10 8

16.5 *(Negative Factor)* If the number in front of the bracket is negative, the signs of the final terms are *opposite* to the signs of the terms inside of the brackets:

$a \quad (b \quad c \quad d) \quad a \quad b \quad a \quad c \quad a \quad d$

Example:

3 (2 4 1)

3 (3) 9

3 2 3 4 3 1 6 12 3 9

1. Complete the following multiplication statements:

a) 20 4 ( ) *b*) 16 (4) ( ) *c*) 10 (5) ( )

d) (3) ( ) 12 *e*) 1 1 ( ) *f*) (1) ( ) 0

g) 100 (10) ( ) *h*) 60 (15) ( ) *i*) 36 (12) ( )

2. Use the associativity property (see 16.2) to calculate easily the product:

a) 2 (3) 5 *b*) (4) (7) (5) *c*) 9 (2) 5

d) (2) 4 (15) 2 *e*) (11) 2 (10) (5) *f*) (25) 3 (5) 4

g) (12) 15 (5) (4) *h*) (25) 6 (5) (4) *i*) (4) (75) (15) 2

3. Use the distributivity property (see 16.3) to rewrite the following expressions by expanding the brackets. Then evaluate the final expression.

a) 2 (3 4) *b*) (3) [(2) (5)] *c*) (2) [(3) (1) (4)]

d) 2 (3 4) *e*) (3) (2 5) *f*) 5 (3 2 4)

g) 1 (5 2) *h*) (2) (3 5) *i*) (4) (2 5 1)

4. Use the distributivity property (see 16.3) to factor out the common factors, then evaluate the final expression. One case is solved for you as an example:

a) 1 2 1 3 *b*) 2 3 2 5 *c*) 3 4 4 5

d) 5 1 5 5 *e*) 10 3 5 10 *f*) 7 1 7 5 7 2

g) 2 1 2 3 2 2 *h*) 1 2 2 3 4 2 *i*) 11 1 11 5 2 11 3 11

j) 10 2 10 3 10 4 *k*) 15 2 15 3 *l*) 25 25 2 25 3 25 4

Example:

h) 1 2 2 3 4 2 2 1 2 3 2 4 2 (1 3 4) 2 (2) 4

5. Evaluate each of the following expressions in two different ways (see 16.3, 16.4 and 16.5):

a) 1 (2 3) *b*) (1) [(2) (3)] *c*) (4) [(1) (2) (3)]

d) 2 (3 4) *e*) (2) (1 3) *f*) 2 (3 1 2)

g) 2 (1 5) *h*) (3) (2 5 1) *i*) (5) (1 2 3)

6. Rewrite one of the factors of multiplication to calculate the product more easily. One case is solved for you as an example:

a) 3 99 *b*) (39) 5 *c*) 102 (6)

d) 51 (4) *e*) (19) (21) *f*) (19) (51)

Example:

f) (19) (51) (20 1) 51 51 (20 1) 51 (20) 51 1
(50 1) (20) 51 50 (20) 1 (20) 51 1000 20 51
1000 31 969

17. Order of Operations (II)

17.1 *(Unary & Binary Operators)* In mathematics, the symbols + and - are used as *unary operators* to express the sign (of a number or of opposite) and as *binary operators* to express the addition and the subtraction operations. Example:

1 (2) (3) (4)

First, second and forth - operator are unary.
Third - operator is binary.
First + operator is binary and second + operator is unary.
The *x* operator is always a binary operator.
The brackets are used here to express the sign of numbers, not to change the order of operations.

17.2 *(Default Order of Operations)* If an expression contains additions, subtractions and multiplications but *not grouping symbols* then the default order of operations is:
a) do all the multiplication operations in the order they appear (from left to right)
b) do all the additions and subtractions in the order they appear (from left to right)
Example:

1 (2) (3) (4)
1 (6) (4) (7) (4)
7 4 3

17.3 *(Order of Operations Algorithm)* If an expression contains *grouping symbols* then use the following algorithm to find the value of the expression:
1. Identify the innermost bracket(s)
2. Perform the operations inside the innermost bracket(s) using the default order of operations
3. Replace the innermost bracket(s) with the result you got on step 2
4. If there are still more grouping symbols then go back to step 1
5. If there are no more grouping symbols, then perform the operations of the remaining expression using the default order of operations.
Example:

3 [2 4 (1 5) 9] (3 4) 5
3 [2 4 (4) 9] (1) 5
3 [2 16 9] (1) 5
3 [9] (1) 5 3 9 5 7

1. For each of the following expressions, analyze each operator and decide if it is an unary or binary operator:

a) (1) (2) (3) (5) *b*) ((5)) 7 6

c) 6 (5) (5) (6) 1 *d*) 5 5 6 3

2. Use the "left to right" rule (see 17.2) to calculate the value of the following expressions containing more than one multiplications:

a) 1 2 3 *b*) 1 (2) 3 *c*) 1 (2) (3)

d) (1) (2) (3) *e*) 2 (1) (3) (2) *f*) (2) (2) (2) (2)

g) 1 (5) (4) 0 *h*) 0 (2) 3 100 *i*) 10 (5) 0 (10)

j) 1 (2) 3 (4) 5 *k*) (6) 4 (2) (1) 2 *l*) 10 (2) 5 (3) (4)

3. Use the default order of operations (see 17.2) to calculate the value of the following expressions (brackets are used here to express the sign of integers):

a) 1 2 3 *b*) 1 2 3 *c*) 1 2 3

d) 1 2 3 *e*) 1 2 3 *f*) 1 (2) 3

g) 1 2 3 4 *h*) 1 2 3 4 *i*) 1 2 3 4

j) 1 2 3 4 *k*) 1 (2) 3 4 *l*) 1 2 3 4

m) 1 2 (3) (4) *n*) (1) (2) 3 (4) 2 *o*) 1 2 3 4 5 2

p) 2 (3) 3 (4) 4 (5) 1 2 (4)

q) 1 2 3 (2) (1) (2) 3 (4) (2) 2 (2) 2 (5)

r) 5 (4) 3 (1) (2) 4 (2) (1) 2 1 2 3 (2) 3 (2) (1)

4. Use the order of operations algorithm (see 17.3) to calculate the value of the following expressions:

a) (1 2) 3 *b*) 1 (2 3) *c*) 1 (2 3)

d) (1 2) 3 *e*) 1 (2 3) *f*) (1 (2) 3)

g) 1 (2 3) 4 *h*) (1 2) 3 4 *i*) 1 (2 3) 4

j) (1 2) (3 4) *k*) 1 (2 3) (4) *l*) 1 (2 3) 4

m) (1 2) (3 4) *n*) ((1) (2 3)) (4) 2 *o*) 1 ((2 3) 4 5) 2

p) (2 (3) 3 (4) 4) (2 1) 2 4

q) (1 2 1) (2) (1 2) 2 (4 2) (2 (2) 3) (1)

r) 2 [(4 3) (1) 2] {4 [(2) (1) (2 1 2) 3 1] 3} (1) 1

s) ((((2 (3) 5) 2) (1) 2) (5) 4) 2 3 (3 2 (3 5 (3 2 2)))

5. Calculate the value of the following expressions:

a) 2 |2 3| *b*) |6 (5)| |(5) (6)| 1 *c*) | 1| | 2| | 3| | 5|

d) |3 5| (3 5) *e*) |(5) (2)| 2 | 3| *f*) | 5 5 6| |3 (2)|

18. Division of Integers (I)

18.1 *(Division)* The *division* operation was initially designed as the *opposite operation of multiplication*. So, a multiplication statement like:

$$a \quad b \quad c$$

is considered equivalent to any one of the following division statements:

$$a \quad \frac{c}{b} \quad or \quad b \quad \frac{c}{a}$$

Example:

$$(\ 3) \quad (\ 4) \qquad 12 \qquad\qquad 3 \quad \frac{12}{4} \qquad 4 \quad \frac{12}{3}$$

18.2 *(Definitions)* The division operation is an operation between a *dividend* and a *divisor* to obtain a result called *quotient*:

$$\frac{divident}{divisor} \quad quotient$$

18.3 *(Sign Rules)* The sign rules for division are similar to the sign rules for multiplication:

$$\frac{a}{b} \quad \frac{a}{b} \qquad \frac{a}{b} \quad \frac{a}{b}$$

$$\frac{a}{b} \quad \frac{a}{b} \qquad \frac{a}{b} \quad \frac{a}{b}$$

Examples:

$$\frac{10}{5} \quad \frac{10}{5} \quad 2 \qquad \frac{15}{3} \quad \frac{15}{3} \quad 5$$

$$\frac{20}{5} \quad \frac{20}{5} \quad 4 \qquad \frac{42}{6} \quad \frac{42}{6} \quad 7$$

18.4 *(Division Operators)* There are four operators used to express the division operation:

$$\frac{a}{b} \qquad a/b \qquad a \quad b \qquad a:b$$

Examples:

$$\frac{8}{4} \quad 2 \qquad (\ 6)/(\ 2) \quad 3$$

$$(\ 10) \quad (\ 2) \quad 5 \qquad (\ 18):(\ 3) \quad 6$$

18.5 *(Division by 1)* Any number divided by *1* is equal to the initial number:

$$\frac{a}{1} \quad a$$

18.6 *(Division by 0)* Division by *0* is not defined.

$$\frac{a}{0} \quad not\ defined$$

1. Change each of the multiplication statements to an equivalent division statement:

a) $2 \quad 1 \quad (\ 1)$ *b)* $9 \quad (\ 3) \quad (\ 3)$ *c)* $8 \quad (\ 2) \quad 4$ *d)* $(\ 1) \quad 1 \quad 1$

e) $5 \quad 2 \quad 10$ *f)* $15 \quad (\ 3) \quad 5$ *g)* $(\ 1) \quad 0 \quad 0$ *h)* $0 \quad 0 \quad 0$

2. Change the division statements to equivalent multiplication statements:

a) $\frac{2}{1} \quad 1$ *b)* $\frac{12}{3} \quad 4$ *c)* $\frac{0}{1} \quad 0$ *d)* $\frac{10}{1} \quad 10$ *e)* $\frac{20}{5} \quad 4$

f) $\frac{8}{4} \quad 2$ *g)* $2 \quad \frac{10}{5}$ *h)* $\frac{6}{3} \quad 2$ *i)* $\frac{0}{2} \quad 0$ *j)* $\frac{100}{10} \quad 10$

3. Find the value of each expression (see 18.3):

a) $\frac{2}{2}$ *b)* $\frac{10}{5}$ *c)* $\frac{0}{2}$ *d)* $\frac{4}{4}$ *e)* $\frac{20}{5}$ *f)* $\frac{18}{6}$

g) $\frac{9}{3}$ *h)* $\frac{16}{4}$ *i)* $\frac{9}{3}$ *j)* $\frac{15}{5}$ *k)* $\frac{12}{6}$ *l)* $\frac{25}{5}$

4. Find the value of each of the following expressions containing different types of division operators (see 18.4):

a) $\frac{3}{1}$ *b)* $6/(\ 2)$ *c)* $(\ 8) \quad (\ 4)$ *d)* $5:(\ 5)$

e) $\frac{9}{3}$ *f)* $(\ 20)/10$ *g)* $8 \quad (\ 2)$ *h)* $25:(\ 5)$

5. Find the value of the following special cases of integer division (see 18.5 and 18.6):

a) $\frac{3}{3}$ *b)* $\frac{3}{1}$ *c)* $\frac{5}{5}$ *d)* $\frac{5}{5}$ *e)* $\frac{1}{1}$ *f)* $\frac{10}{10}$

g) $\frac{11}{11}$ *h)* $\frac{5}{1}$ *i)* $\frac{2}{0}$ *j)* $\frac{6}{1}$ *k)* $\frac{0}{7}$ *l)* $\frac{0}{0}$

6. Find the value of each expression:

a) $\frac{28}{7}$ *b)* $\frac{36}{12}$ *c)* $\frac{45}{5}$ *d)* $\frac{50}{10}$ *e)* $\frac{75}{25}$ *f)* $\frac{100}{50}$

g) $\frac{121}{11}$ *h)* $\frac{250}{50}$ *i)* $\frac{200}{40}$ *j)* $\frac{600}{150}$ *k)* $\frac{1000}{125}$ *l)* $\frac{625}{25}$

7. Find the value of each expression containing the absolute value function. One case is solved for you as an example:

a) $\frac{|\ 3|}{3}$ *b)* $\frac{5}{|\ 5|}$ *c)* $\frac{|\ 15|}{|\ 5|}$ *d)* $\frac{|\ 12|}{4}$ *e)* $\frac{18}{|\ 3|}$

f) $\left|\frac{20}{4}\right|$ *g)* $\left|\frac{|\ 10|}{5}\right|$ *h)* $\frac{|0|}{|\ 5|}$ *i)* $\left|\frac{0}{11}\right|$ *j)* $\left|\frac{15}{5}\right|$

Example:

g) $\left|\frac{|\ 10|}{5}\right| \quad \left|\frac{10}{5}\right| \quad |\ 2| \quad 2$

19. Division of Integers (II)

19.1 *(Rational Numbers)* The set of integers **Z** is *not closed* under the operation of division. This means that *the quotient of two integers might not be an integer*. In these cases the quotients of two integers define more complex numbers in mathematics than integers called *rational numbers*. Example:

$$\frac{7}{3}$$

19.2 *(Lowest Terms)* The quotient of two integers can be simplified by dividing both the dividend and the divisor by *common factors* until you get the *quotient in lowest terms*. Example:

$$\frac{24}{18} \quad \frac{(\;24)\;\;3}{(\;18)\;\;3} \quad \frac{8}{6} \quad \frac{(\;8)\;\;2}{(\;6)\;\;2} \quad \frac{4}{3}$$

19.3 *(Signed Decimals)* If the quotient of two integers is not an integer, you can express the quotient using a *signed terminating or non-terminating decimal*.

$$\frac{13}{5} \quad 2.6 \qquad \frac{5}{3} \quad 2.666... \quad 2.\overline{6}$$

19.4 *(Multiples)* Given an integer, you can find its multiples by multiplying the integer by *+1*, *-1*, *+2*, *-2*, *+3*, *-3*, ...
So, the multiples of *-3* are:

3, 3, 6, 6, 9, 9, 12, 12,...

19.5 *(Exact Division)* If the dividend is a multiple of the divisor, the quotient is an integer. In this case the dividend is divided exactly by the divisor. Example:

$$\frac{12}{3} \quad 4$$

19.6 *(Remainder)* If the dividend is not a multiple of the divisor, the division has a quotient and a remainder. The *quotient* is an integer and the *remainder* is a natural number less than the absolute value of the divisor. Here is the *division statement* and some examples:

$$\frac{D}{d} \quad q \quad \frac{r}{d} \quad \text{or} \quad D \quad q \quad d \quad r; \quad 0 \quad r \quad |d|$$

$$\frac{9}{2} \quad 4R1 \quad \text{because} \quad 9 \quad 4 \quad 2 \quad 1$$

$$\frac{9}{2} \quad 4R1 \quad \text{because} \quad 9 \quad (\;4) \quad (\;2) \quad 1$$

$$\frac{9}{2} \quad 5R1 \quad \text{because} \quad 9 \quad (\;5) \quad 2 \quad 1$$

$$\frac{9}{2} \quad 5R1 \quad \text{because} \quad 9 \quad 5 \quad (\;2) \quad 1$$

1. Write the following ratios of integers as rational numbers in lowest terms:

$a)\ \frac{3}{4} \quad b)\ \frac{14}{4} \quad c)\ \frac{16}{6} \quad d)\ \frac{6}{4} \quad e)\ \frac{20}{15} \quad f)\ \frac{32}{18}$

$g)\ \frac{8}{6} \quad h)\ \frac{20}{12} \quad i)\ \frac{30}{12} \quad j)\ \frac{25}{10} \quad k)\ \frac{24}{16} \quad l)\ \frac{18}{16}$

2. Express the following ratios of integers as signed terminating decimals (see 19.3):

$a)\ \frac{1}{4} \quad b)\ \frac{2}{5} \quad c)\ \frac{3}{12} \quad d)\ \frac{5}{8} \quad e)\ \frac{7}{10} \quad f)\ \frac{2}{10}$

$g)\ \frac{3}{15} \quad h)\ \frac{10}{25} \quad i)\ \frac{6}{24} \quad j)\ \frac{35}{10} \quad k)\ \frac{24}{16} \quad l)\ \frac{5}{20}$

3. Express the following ratios of integers as signed non-terminating repeating decimals (see 19.3):

$a)\ \frac{1}{3} \quad b)\ \frac{4}{6} \quad c)\ \frac{4}{12} \quad d)\ \frac{20}{15} \quad e)\ \frac{2}{12}$

$f)\ \frac{4}{44} \quad g)\ \frac{2}{6} \quad h)\ \frac{5}{15} \quad i)\ \frac{8}{24} \quad j)\ \frac{15}{90}$

4. For each given integer, find 5 positive and 5 negative multiples:

$a)\ 2 \quad b)\ 5 \quad c)\ 6 \quad d)\ 1 \quad e)\ 7$

$f)\ 2 \quad g)\ 5 \quad h)\ 4 \quad i)\ 8 \quad j)\ 10$

5. Rewrite each of the following division statements:

$a)\ \frac{7}{3} \quad 2R1 \quad b)\ \frac{9}{4} \quad 2R1 \quad c)\ \frac{9}{3} \quad 3R0 \quad d)\ \frac{11}{4} \quad 2R3$

$e)\ \frac{10}{3} \quad 3R1 \quad f)\ \frac{17}{3} \quad 5R2 \quad g)\ \frac{19}{5} \quad 3R4 \quad h)\ \frac{21}{2} \quad 10R1$

6. Find the quotient and the remainder for each case:

$a)\ \frac{5}{2} \quad b)\ \frac{7}{4} \quad c)\ \frac{9}{5} \quad d)\ \frac{11}{3}$

$e)\ \frac{15}{7} \quad f)\ \frac{15}{4} \quad g)\ \frac{15}{3} \quad h)\ \frac{15}{2}$

7. Rewrite each of the following division statements (see 19.6):

$a)\ \frac{9}{4} \quad 2R1 \quad b)\ \frac{9}{4} \quad 2R1 \quad c)\ \frac{9}{4} \quad 3R3 \quad d)\ \frac{9}{4} \quad 3R3$

$e)\ \frac{17}{5} \quad 3R2 \quad f)\ \frac{17}{5} \quad 3R2 \quad g)\ \frac{17}{5} \quad 4R3 \quad h)\ \frac{17}{5} \quad 4R3$

8. Find the quotient and the remainder for each case (see 19.6):

$a)\ \frac{7}{3} \quad b)\ \frac{7}{3} \quad c)\ \frac{7}{3} \quad d)\ \frac{7}{3}$

$e)\ \frac{15}{4} \quad f)\ \frac{15}{4} \quad g)\ \frac{15}{4} \quad h)\ \frac{15}{4}$

20. Order of Operations (III)

20.1 *(Consecutive Divisions)* If more divisions occur consecutively within an expression, the divisions must be done one by one *in the order that they occur* (from left to right). Example:

(24) (6) (2) (4) (2) 2

20.2 *(Brackets)* Use brackets to specify a particular order of operations. Example:

(24) [(6) (2)] (24) (3) 8

20.3 *(Consecutive Multiplications and Divisions)* Multiplications and divisions are considered operations of *equal priority*. If multiplications and divisions appear consecutively, perform the operations in the order that they occur (from left to right). Brackets can be used to specify a different order. Examples:

(6) (4) (12) 3

(24) (12) 3 2 3 6

(6) [(12) (4)] 2

(6) (3) 2 2 2 4

20.4 *(Default Order of Operations, Updated)* Multiplication and division are considered to have *higher priority* than addition and subtraction. If an expression contains addition, subtraction, multiplication and division but *not grouping symbols*, then the default order of operations is:
a) do all the multiplication and division operations in the order that they appear (from left to right)
b) do all the addition and subtraction operations in the order that they appear (from left to right). Example:

2 3 (6) 3 3 (2)

6 (2) (6) 2

20.5 *(Order of Operations Algorithm)* If an expression contains *grouping symbols*, use the Order of Operations Algorithm presented in 17.3 and the Default Order of Operations presented above (20.4). Example:

2 [3 (1 2 3) (4 2) 6 (3) 1]

(2 3 12 (4)) (10) 2

2 [3 (7) (2) (2) 1]

(6 (3)) (5)

2 [3 (14) 2 1] 9 5

2 [14] 4 28 4 24

1. Use the "left to right" order of operations to find the value of each of the following expressions (see 20.1):

a) 12 (2) 2 *b)* 20 (2) 5 *c)* 50 (5) 2

d) 24 2 (3) 2 *e)* 64 2 (2) 2 *f)* 32 (2) (2) (1)

g) 40 (1) (2) (5) *h)* 75 (3) (1) 5 *i)* 36 2 (3) (2) (1)

2. Use the "left to right" order of operations to find the value of each of the following expressions (see 20.3):

a) 4 (3) (2) *b)* 10 (5) 2 *c)* 5 (1) (5) 2

d) 12 2 (1) 3 *e)* 16 2 (2) 2 *f)* 5 (1) (1) (1)

g) 3 (6) (3) 2 *h)* 2 (3) (2) 3 *i)* 1 (4) (6) (8) (2)

3. Start with the inner-most brackets and find the value of each expression (see 20.2):

a) 16 [(8) (2)] *b)* 20 [(5) 4] *c)* [8 (3)] [(4) (2)]

d) $\frac{2\ \ 8}{4}\ (\ 2)$ *e)* $12\ \frac{16}{4\ (\ 2)}$ *f)* $\frac{(\ 1)\ (\ 12)}{2\ (\ 2)}\ \frac{16\ (\ 4)}{4}$

g) $\frac{(\ 3)\ (\ 4)\ 6}{20\ [5\ (\ 2)]}$ *h)* $\frac{(\ 3)\ 2\ (\ 2)}{4\ 9\ (\ 3\ 3)}$ *i)* $\frac{(\ 12)\ (\ 3)}{24\ (\ 12)}\ \frac{10\ (\ 5)}{25\ (\ 5)}$

4. Use the "default order of operations" to find the value of each of the following expressions (see 20.4):

a) 1 2 3 4 (2) *b)* 12 4 3 (2) *c)* 10 2 (3) (1) 5

d) 1 10 (2) 3 1 *e)* 15 (5) 3 (2) 5 *f)* 4 5 20 (5) 4 3

g) 24 (12) 3 (1) *h)* 2 (8) 4 2 (3) *i)* 1 (1) (1) (1) 1

5. Use the "order of operations algorithm" (see 20.5) to find the value of each of the following expressions:

a) (1 2 3) (5) 1 *b)* 12 (3 3 3) 2 *c)* [3 (2) 4] (2 (2) 1)

d) $1\ \frac{3\ 2\ 4}{5}\ (\ 2)$ *e)* $2\ \frac{12\ (\ 3)\ 1}{(\ 1)\ (\ 2)\ 1}$ *f)* $\frac{2\ 2\ (\ 6)}{2\ (\ 2)\ 1}\ \frac{3\ 16\ (\ 4)}{3\ 2\ (\ 2)}$

g) 1 [2 (3 4)] (3 5) [(2 (6) 2) 5] *h)* $(\ 2)\ \frac{1\ 2\ (\ 3)\ 6}{15\ (\ 5)\ 2}\ 3$

i) [15 (7 2)] [5 4 (2)] (4 10 2) *j)* $\frac{8}{3\ 5}\ \frac{4\ 20\ (\ 5)}{2\ 3\ 2}\ (\ 2)$

k) 2 (2 3 8 4) (10 2 2 2) (9 (3) 1) (3 (5) 4 5 2) 5

l) $\frac{4\ 5\ 2}{2}\ \frac{(\ 2\ 2\ 3\ 1)\ 1}{(\ 9)\ (\ 3)}\ \frac{(4\ 5\ 5)\ 3}{1\ 9\ (\ 3)}\ \frac{(15\ 3\ 1)\ 3}{1\ 2}\ \frac{2\ 5\ 8\ 2}{2\ (\ 5)\ 3}$

m) $\frac{\frac{(\ 3)\ 4\ 2}{2}}{10\ 2}\ \frac{5\ 4}{\frac{3\ 5\ 1}{2\ (\ 2)}}\ \frac{10}{2}\ \frac{\frac{3\ (\ 5)\ 3}{3}}{2\ (\ 2)}\ \frac{4\ 4\ 2}{\frac{2\ 5\ 1}{3}}\ \frac{15}{5}\ [3\ (\ 3)]$

21. Equalities and equations

21.1 *(Multiplying and Dividing)* By multiplying (or dividing) the left side and the right side of an equality by the same number, you get an *equivalent equality*:

$$a \quad b \qquad a \quad c \quad b \quad c \;;\; c \quad 0$$

$$a \quad b \qquad \frac{a}{c} \quad \frac{b}{c} \;;\; c \quad 0$$

Examples:

$$2 \quad 4 \quad (\ 2) \ (\ 3) \quad (\ 4) \ (\ 3)$$

$$6 \quad 12;\ \textit{false}$$

$$3 \quad 9 \quad 6 \quad \frac{3}{3} \quad \frac{9 \quad 6}{3} \quad 1 \quad 1;\ \textit{true}$$

21.2 *(Unknown Factor)* Solution 1. Rewrite the multiplication as an equivalent division:

$$a \quad x \quad b \qquad x \quad \frac{b}{a}$$

$$2 \quad x \quad 8 \qquad (\ 2) \quad x \quad 8 \qquad x \quad \frac{8}{2} \quad 4$$

Solution 2. Divide the equality by the second factor:

$$a \quad x \quad b \qquad \frac{a \quad x}{a} \quad \frac{b}{a} \qquad x \quad \frac{b}{a}$$

$$(\ 3) \quad x \quad 18; \frac{(\ 3) \quad x}{(\ 3)} \quad \frac{18}{3}; x \quad \frac{18}{3} \quad 6$$

21.3 *(Unknown Dividend)* Solution 1. Rewrite the division as an equivalent multiplication:

$$\frac{x}{a} \quad b \qquad x \quad a \quad b$$

$$\frac{x}{4} \quad 3 \qquad x \quad (\ 4) \quad (\ 3) \quad 12$$

Solution 2. Multiply the equation by the divisor:

$$\frac{x}{a} \quad b \qquad a \quad \frac{x}{a} \quad a \quad b \qquad x \quad a \quad b$$

$$\frac{x}{2} \quad 9; (\ 2) \quad \frac{x}{(\ 2)} \quad (\ 2) \quad (\ 9);$$

$$x \quad (\ 2) \quad (\ 9) \quad 18$$

21.4 *(Unknown Divisor)* Solution 1. Rewrite the division as an equivalent multiplication and then back as an equivalent division:

$$\frac{a}{x} \quad b \qquad a \quad b \quad x \qquad x \quad \frac{a}{b}$$

$$\frac{24}{x} \quad 6 \qquad 24 \quad (\ 6) \quad x \qquad x \quad \frac{24}{6} \quad 4$$

Solution 2. Multiply the equation by the divisor and divide the equation by the initial quotient:

$$\frac{a}{x} \quad b; \frac{a}{x} \quad x \quad b \quad x; a \quad b \quad x; \frac{a}{b} \quad \frac{b \quad x}{b}; x \quad \frac{a}{b}$$

$$\frac{8}{x} \quad 4; \frac{8}{x} \quad x \quad 4 \quad x; 8 \quad 4 \quad x; \frac{8}{4} \quad \frac{4 \quad x}{4}; x \quad 2$$

1. All the following equalities are true statements. Multiply or divide both sides of each equality by the given integer and check if the statement remains true (see 21.1).

a) $1 \quad 2 \quad 3;$ *multiply by* 3 *b)* $2 \quad 2 \quad (\ 3) \quad 4;$ *divide by* 2

c) $4 \quad (5 \quad 10) \quad (\ 5) \quad 3;$ *multiply by* 1 *d)* $4 \quad (\ 3) \quad 5 \quad (\ 3) \quad 3;$ *divide by* 6

2. Solve each of the following equations for x by replacing each multiplication statement with an equivalent division statement (see 21.2):

a) $3 \quad x \quad 15$ *b)* $x \quad (\ 4) \quad 8$ *c)* $4 \quad x \quad 12$ *d)* $x \quad 7 \quad 14$

e) $10 \quad 2 \quad x$ *f)* $9 \quad x \quad (\ 3)$ *g)* $20 \quad 5 \quad x$ *h)* $0 \quad x \quad (\ 3)$

3. Solve each of the following equations for x by dividing both sides by a convenient integer (see 21.2):

a) $1 \quad x \quad 4$ *b)* $x \quad (\ 2) \quad 12$ *c)* $3 \quad x \quad 15$ *d)* $x \quad (\ 4) \quad 16$

e) $21 \quad 3 \quad x$ *f)* $18 \quad x \quad (\ 3)$ *g)* $25 \quad 5 \quad x$ *h)* $0 \quad x \quad (\ 5)$

4. Solve each of the following equations for x by replacing the division statement with an equivalent multiplication statement (see 21.3):

a) $\frac{x}{3} \quad 4$ *b)* $\frac{x}{5} \quad 4$ *c)* $\frac{x}{4} \quad 4$ *d)* $\frac{x}{10} \quad 0$

e) $5 \quad \frac{x}{6}$ *f)* $1 \quad \frac{x}{7}$ *g)* $7 \quad \frac{x}{3}$ *h)* $1 \quad \frac{x}{1}$

5. Solve each of the following equations for x by multiplying both sides by a convenient integer (see 21.3):

a) $\frac{x}{1} \quad 3$ *b)* $\frac{x}{3} \quad 4$ *c)* $\frac{x}{2} \quad 6$ *d)* $\frac{x}{3} \quad 0$

e) $3 \quad \frac{x}{2}$ *f)* $4 \quad \frac{x}{3}$ *g)* $5 \quad \frac{x}{2}$ *h)* $3 \quad \frac{x}{3}$

6. In order to solve each of the following equations for x, rewrite the division statement as a multiplication statement and then back as a division statement (see 21.4):

a) $\frac{5}{x} \quad 1$ *b)* $\frac{15}{x} \quad 3$ *c)* $\frac{25}{x} \quad 5$ *d)* $\frac{24}{x} \quad 6$

e) $3 \quad \frac{18}{x}$ *f)* $7 \quad \frac{21}{x}$ *g)* $5 \quad \frac{50}{x}$ *h)* $9 \quad \frac{36}{x}$

i) $1 \quad \frac{1}{x}$ *j)* $2 \quad \frac{8}{x}$ *k)* $5 \quad \frac{25}{x}$ *l)* $7 \quad \frac{28}{x}$

7. In order to solve each of the following equations for x, multiply and then divide both sides by convenient integers (see 21.4):

a) $\frac{1}{x} \quad 1$ *b)* $\frac{35}{x} \quad 7$ *c)* $\frac{100}{x} \quad 20$ *d)* $\frac{30}{x} \quad 5$

e) $2 \quad \frac{16}{x}$ *f)* $11 \quad \frac{33}{x}$ *g)* $7 \quad \frac{21}{x}$ *h)* $8 \quad \frac{40}{x}$

i) $6 \quad \frac{30}{x}$ *j)* $9 \quad \frac{36}{x}$ *k)* $3 \quad \frac{300}{x}$ *l)* $1 \quad \frac{13}{x}$

22. Proportions and Equations

22.1 *(Proportions)* A proportion is an equation with a ratio on each side:

$$\frac{a}{b} \quad \frac{c}{d}$$

22.2 *(Solving a proportion)* To solve a proportion means to find one *term* of the proportion when three other *terms* are given. Use *cross-multiplication* and *division* to isolate the unknown term:

$$\frac{a}{x} \quad \frac{b}{c}; a \quad c \quad b \quad x; \frac{a \quad c}{b} \quad \frac{b \quad x}{b}; x \quad \frac{a \quad c}{b}$$

Example:

$$\frac{5}{3} \quad \frac{30}{x}; 5 \quad x \quad 90; \frac{5 \quad x}{5} \quad \frac{90}{5}; x \quad 14$$

22.3 *(Shortcut for a proportion)* To find a *term* of a proportion multiply the adjacent *terms* and divide by the opposite *term*:

$$\frac{a}{b} \quad \frac{c}{d}$$

$$a \quad \frac{b \quad c}{d}; b \quad \frac{a \quad d}{c}; c \quad \frac{a \quad d}{b}; d \quad \frac{b \quad c}{a}$$

Example:

$$\frac{12}{3} \quad \frac{x}{5} \quad x \quad \frac{(\;12) \quad 5}{3} \quad \frac{60}{3} \quad 20$$

22.4 *(Other equations)* The main idea in solving an equation is to isolate the variable using equivalent equations.

Example 1:

$$3 \quad 2 \quad (x \quad 5) \quad 11 \qquad 2 \quad (x \quad 5) \quad 11 \quad 3$$

$$2 \quad (x \quad 5) \quad 8 \qquad x \quad 5 \quad \frac{8}{2}$$

$$x \quad 5 \quad 4 \qquad x \quad 4 \quad 5 \quad 9$$

Example 2:

$$5 \quad 10 \quad x \quad (\;2) \quad 3 \quad 4$$

$$10 \quad x \quad (\;2) \quad 4 \quad 5 \quad 3$$

$$10 \quad x \quad (\;2) \quad 4 \qquad 10 \quad x \quad \frac{4}{2}$$

$$10 \quad x \quad 2 \qquad \frac{10}{x} \quad \frac{2}{1} \qquad x \quad \frac{10}{2} \quad 5$$

Example 3:

$$\frac{3 \quad 15}{\frac{8}{x} \quad 2} \quad \frac{9}{3} \qquad \frac{12}{\frac{8}{x} \quad 2} \quad \frac{3}{1}$$

$$\frac{8}{x} \quad 2 \quad \frac{12}{3} \qquad \frac{8}{x} \quad 4 \quad 2$$

$$\frac{8}{x} \quad \frac{2}{1} \qquad x \quad \frac{8}{2} \quad 4$$

1. Use cross-multiplication and division to solve for x the following equations (see 22.2):

$a)\ \frac{x}{2} \quad \frac{4}{8}$ $\quad b)\ \frac{5}{x} \quad \frac{2}{4}$ $\quad c)\ \frac{3}{12} \quad \frac{x}{8}$ $\quad d)\ \frac{0}{10} \quad \frac{x}{2}$

$e)\ \frac{x}{4} \quad \frac{10}{8}$ $\quad f)\ \frac{3}{x} \quad \frac{9}{6}$ $\quad g)\ \frac{4}{2} \quad \frac{x}{8}$ $\quad h)\ \frac{1}{1} \quad \frac{1}{x}$

2. Use cross-multiplication and division (the shortcut method explained at 22.3) to solve for x the following equations:

$a)\ \frac{x}{1} \quad \frac{5}{5}$ $\quad b)\ \frac{15}{x} \quad \frac{3}{2}$ $\quad c)\ \frac{2}{3} \quad \frac{x}{6}$ $\quad d)\ \frac{0}{1} \quad \frac{x}{5}$

$e)\ \frac{x}{7} \quad \frac{4}{14}$ $\quad f)\ \frac{8}{x} \quad \frac{2}{3}$ $\quad g)\ \frac{5}{4} \quad \frac{x}{20}$ $\quad h)\ \frac{15}{10} \quad \frac{3}{x}$

3. Use equivalent equations to solve for x the following equations:

$a)\ 2 \quad x \quad 3 \quad 9$ $\qquad b)\ 4 \quad 3 \quad x \quad 5$ $\qquad c)\ 12 \quad 5 \quad x \quad 2$

$d)\ x \quad 2 \quad 2 \quad 2$ $\qquad e)\ 6 \quad x \quad 4 \quad 1$ $\qquad f)\ 5 \quad 2 \quad x \quad (\;5)$

$g)\ \frac{x \quad 3}{2} \quad \frac{4}{2}$ $\qquad h)\ \frac{5}{2 \quad x \quad 4} \quad \frac{2}{4}$ $\qquad i)\ \frac{6}{10} \quad \frac{3 \quad 5 \quad x}{5}$

$j)\ \frac{x \quad 3 \quad 1}{3} \quad \frac{6}{9}$ $\qquad k)\ \frac{4}{6 \quad x \quad 1} \quad \frac{6}{3}$ $\qquad l)\ \frac{20}{5} \quad \frac{8}{x \quad 5 \quad 2}$

$m)\ 3 \quad (x \quad 1) \quad 1 \quad 2$ $\quad n)\ 2 \quad (x \quad 3) \quad 5 \quad 3$

$o)\ \frac{5 \quad 2 \quad (x \quad 2)}{6} \quad \frac{5}{10}$ $\quad p)\ \frac{3}{1} \quad \frac{18}{1 \quad (x \quad 2 \quad 3)}$

$q)\ [5 \quad (2 \quad x \quad 3) \quad (\;3)] \quad [\;12 \quad 4 \quad 2] \quad 2 \quad (3 \quad 1 \quad 3 \quad 2) \quad 3$

$r)\ \frac{3 \quad (\;4) \quad 2}{2 \quad 10 \quad (\;2)} \quad \frac{3 \quad (\;5) \quad 1}{\frac{10}{x} \quad 4 \quad (\;2 \quad 3) \quad 1}$

4. Solve for x using the "working backward" method. One case is solved for you as an example.

$a)\ 10 \quad x \quad (\;4) \quad 8$ $\qquad b)\ 5 \quad x \quad (\;4) \quad 5$

$c)\ 12 \quad x \quad 5 \quad 4 \quad 5 \quad 0$ $\qquad d)\ 5 \quad (\;4) \quad x \quad 15 \quad 4 \quad 0$

$e)\ (\;1) \quad 2 \quad x \quad (\;4) \quad 32$ $\qquad f)\ (x \quad 3) \quad (\;18 \quad 3) \quad (\;7) \quad 6 \quad 0$

$g)\ 36 \quad (\;2) \quad x \quad (\;2) \quad 1$ $\qquad h)\ \frac{2}{10 \quad x \quad 5 \quad 3} \quad \frac{3}{3}$

Example:

$a)\ 10 \quad x \quad (\;4) \quad 8 \quad 10 \quad x \quad 8 \quad (\;4) \quad 10 \quad x \quad 2 \quad \frac{10}{x} \quad \frac{2}{1}$

$x \quad \frac{10 \quad 1}{2} \quad 5$

23. Inequalities and Inequations (II)

23.1 *(Positive Numbers)* By multiplying (or dividing) an inequality (or inequation) by a *positive number* you get an equivalent inequality (or inequation).
Example 1 (all statements are false):

$$5 \quad 10 \quad \frac{5}{5} \quad \frac{10}{5} \quad 1 \quad 2$$

Example 2 (all statements are true):

$$3 \quad 4; (\ 2) \ (\ 3) \quad (\ 2) \ (\ 4); \quad 6 \quad 8$$

23.2 *(Negative Numbers)* By multiplying (or dividing) an inequality (or inequation) by a *negative number* you get an equivalent inequality (or inequation) only if you *change its sense*. Examples (all are true):

$$1 \quad 2 \quad (\ 3) \ 1 \quad (\ 3) \ 2 \quad 3 \quad 6$$

$$9 \quad 12 \quad \frac{9}{3} \quad \frac{12}{3} \quad 3 \quad 4$$

23.3 *(Inequations)* The main idea in solving an inequation is to write a sequence of equivalent inequations until you can isolate the variable. Examples:

1. $2 \ x \quad 8 \quad \frac{2 \ x}{2} \quad \frac{8}{2} \quad x \quad 4$

2. $\frac{x}{5} \quad 3; (\ 5) \ \frac{x}{5} \quad (\ 5) \ (\ 3);$

$x \quad 15; (\ 1) \ (\ x) \quad (\ 1) \ (\ 15); x \quad 15$

23.4 *(Inequations with Ratios)* If you multiply an inequation by an expression containing the variable, you have to separately analyze the cases when the expression is positive or negative.

*Ex.*1. $\frac{2}{x} \quad \frac{3}{9}; (\ 9) \ \frac{2}{x} \quad (\ 9) \ \frac{3}{9};$

$\frac{18}{x} \quad 3; \frac{1}{3} \ \frac{18}{x} \quad \frac{3}{3}; \frac{6}{x} \quad 1$

If $x \quad 0$ *then* $6 \quad x; x \quad 6$ (*no solution*)

If $x \quad 0$ *then* $6 \quad x; x \quad 6;$

$x \quad 0$ *and* $x \quad 6 \quad x \quad \{\ 6,\ 5,\ 4,\ 3,\ 2,\ 1\}$

*Ex*2. $2 \quad \frac{6}{x \ 1}$

Case 1. $x \ 1 \ 0 \quad x \ 1$

$(\ x \ 1) \ (\ 2) \quad (\ x \ 1) \ \frac{6}{x \ 1};$

$(\ x \ 1) \ (\ 2) \quad 6; \ x \ 1 \quad 3; x \quad 4$

$x \quad 1$ *and* $x \quad 4$ *no solution*

Case 2. $x \ 1 \ 0 \quad x \ 1$

$(\ x \ 1) \ (\ 2) \quad (\ x \ 1) \ \frac{6}{x \ 1};$

$(\ x \ 1) \ (\ 2) \quad 6; \ x \ 1 \quad 3; x \quad 4$

$x \quad 1$ *and* $x \quad 4 \quad x \quad \{\ 2,\ 3\}$

1. All of the following inequalities are true statements. Multiply or divide both sides of each inequality by the given positive integer and check if the statement remains true (see 23.1).

a) $1 \quad 2 \quad 3$; *multiply by* 2 *b*) $6 \quad 2 \quad (\ 1) \quad 5$; *divide by* 3

c) $0 \quad 12 \quad (\ 4) \quad 3$; *multiply by* 3 *d*) $4 \quad (\ 3) \quad (\ 6) \quad 4$; *divide by* 2

2. All of the following inequalities are true statements. Multiply or divide both sides of each inequality by the given negative integer and change the sense of the inequality. Check if the statement remains true (see 23.2).

a) $2 \quad 3 \quad 4$; *multiply by* 2 *b*) $5 \quad 2 \quad (\ 10) \quad 15$; *divide by* 5

c) $0 \quad 10 \quad (\ 5) \quad 1$; *multiply by* 3 *d*) $2 \quad (\ 3) \quad (\ 3) \quad 2$; *divide by* 2

3. Solve the following inequations for x by multiplying or dividing both sides by a convenient integer (see 23.3). Look only for integral solutions of x (that is , solutions where x is an integer).

a) $3 \ x \quad 9$ *b*) $2 \ x \quad 10$ *c*) $6 \quad 3 \ x$ *d*) $8 \quad 4 \ x$

e) $\frac{x}{2} \quad 1$ *f*) $\frac{x}{5} \quad 2$ *g*) $5 \quad \frac{x}{2}$ *h*) $0 \quad \frac{x}{3}$

i) $1 \quad \frac{x}{1}$ *j*) $0 \quad \frac{x}{1}$ *k*) $1 \quad \frac{x}{5}$ *l*) $10 \quad \frac{x}{1}$

4. Solve the following inequations for x. Find only the integral solutions of x (x is an integer). An analysis of the sign of x might be necessary (see 23.4).

a) $\frac{1}{x} \quad 0$ *b*) $\frac{2}{x} \quad 0$ *c*) $\frac{2}{x} \quad 2$ *d*) $1 \quad \frac{5}{x}$

e) $\frac{6}{x} \quad 2$ *f*) $\frac{2}{x} \quad 1$ *g*) $2 \quad \frac{6}{x}$ *h*) $5 \quad \frac{10}{x}$

i) $1 \quad \frac{1}{x}$ *j*) $2 \quad \frac{6}{x}$ *k*) $1 \quad \frac{7}{x}$ *l*) $25 \quad \frac{100}{x}$

5. Solve the following inequations for x by isolating the variable (see 23.3). Look only for integral solutions of x (x is an integer).

a) $2 \quad (x \quad 2) \quad 8$ *b*) $3 \quad (2 \quad x \quad 3) \quad 9$ *c*) $10 \quad 5 \quad (\ x \quad 2)$

d) $8 \quad 3 \quad (x \quad 1) \quad 2$ *e*) $\frac{2 \quad (x \quad 1)}{3} \quad 4$ *f*) $5 \quad \frac{(x \quad 1) \quad 2 \quad 3}{3}$

g) $2 \quad \frac{8}{x \quad 3}$ *h*) $2 \quad \frac{10}{3 \quad x \quad 1}$ *i*) $0 \quad \frac{1}{3 \quad (2 \quad x \quad 6)}$

6. Solve the following inequations for x. Look only for integral solutions of x (x is an integer). One case is solved for you as an example:

a) $|x| \quad 2$ *b*) $|x \quad 1| \quad 0$ *c*) $|2 \quad x| \quad 10$ *d*) $|x \quad 2| \quad 2$

e) $|x| \quad 2$ *f*) $|x \quad 1| \quad 0$ *g*) $|2 \quad x| \quad 10$ *h*) $|2 \quad x| \quad 0$

i) $|x| \quad 0$ *j*) $|x| \quad 1 \quad 5$ *k*) $3 \quad |x \quad 1| \quad 3$ *l*) $|\frac{x}{2} \quad 1| \quad 3$

Example:

d) $|x \quad 2| \quad 2$

Case 1. $x \quad 2 \quad 0 \quad x \quad 2 \quad x \quad 2 \quad 2 \quad x \quad 2 \quad 2 \quad x \quad 4$

Case 2. $x \quad 2 \quad 0 \quad x \quad 2 \quad 2 \quad x \quad 2 \quad 2 \quad 2 \quad x \quad 0 \quad x \quad x \quad 0$

24. Powers

24.1 *(Shortcut)* Powers were introduced as shortcuts to *repetitive multiplications* of a number by itself:

$$a^n \quad \underbrace{a \quad a \quad a \quad \ldots \quad a}_{n\ times}$$

Example: $(\ 2) \ (\ 2) \ (\ 2) \quad (\ 2)^3$

24.2 *(Expanded and exponential notations)*
The expanded notation is: $a \quad a \quad a \quad \ldots \quad a$
The exponential notation is: a^n
where a is called the *base* and n is called the *exponent*.
The full expression a^n represents a *power.*
Examples:

1. $(\ 3)^4 \quad (\ 3) \ (\ 3) \ (\ 3) \ (\ 3)$
2. $(\ 5) \ (\ 5) \quad (\ 5)^2$

24.3 *(Negative exponents)* A power with a negative exponent is defined as:

$$a^{\ n} \quad \frac{1}{a^n} \quad \frac{1}{\underbrace{a \quad a \quad a \quad \ldots \quad a}_{n\ times}}$$

Example:

$$(\ 2)^{\ 3} \quad \frac{1}{(\ 2)^3} \quad \frac{1}{(\ 2) \ (\ 2) \ (\ 2)} \quad \frac{1}{8}$$

As you can see from the previous example, an integer raised to a negative integral exponent is a rational number.

24.4 *(Base 0 and 1) 0* cannot be a base for powers having integral negative exponents because division by *0* is not defined. Also *1* can be a base for any integral exponents. This operation doesn't have an inverse operation (see sections about Roots).

$$0^4 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$$
$$1^3 \quad 1 \quad 1 \quad 1 \quad 1$$

24.5 *(Exponent 0 and 1)* Any number (excluding 0) raised to power of 0 is 1. Any number raised to power of 1 is equal to itself:

$$a^0 \quad 1 \qquad a^1 \quad a$$

24.5 *(Power Operator)* The power is a *binary operation* between a *base* and an *exponent*. Usually the base is written normally and the exponent is written as superscript. Sometimes the ^ operator is used to denote a power operation, especially on calculators and in programing languages:

$$a^n \qquad a \wedge n$$

Example:

$$(\ 3)\wedge(\ 2) \quad \frac{1}{(\ 3) \ (\ 3)} \quad \frac{1}{9}$$

1. Change the following repetitive multiplications to powers (see 24.1):

a) 3 3 3 3 3 3 3 *b*) (5) (5) (5) (5)

c) (4) *d*) (10) (10) (10) (10) (10)

e) (2) (2) (2) *f*) (1) (1) (1) (1) (1) (1)

2. Use expanded notation to express the following powers (see 24.2):

a) 3^4 *b*) $(\ 2)^3$ *c*) $(\ 1)^2$ *d*) $(\ 3)^3$ *e*) $(\ 2)^4$ *f*) $(\ 8)^2$

g) 10^3 *h*) $(\ 10)^2$ *i*) $(\ 5)^3$ *j*) 1^5 *k*) $(\ 7)^2$ *l*) $(\ 6)^3$

3. Use expanded notation to express the following powers (see 24.3):

a) $4^{\ 2}$ *b*) $(\ 2)^{\ 1}$ *c*) $(\ 1)^{\ 3}$ *d*) $5^{\ 3}$ *e*) $(\ 3)^{\ 2}$

f) $10^{\ 2}$ *g*) $(\ 10)^{\ 3}$ *h*) $(\ 4)^{\ 2}$ *i*) $1^{\ 5}$ *j*) $(\ 5)^{\ 4}$

4. Use expanded notation to find the value of the following expressions. One case is solved for you as an example.

a) 2^3 *b*) $2^{\ 3}$ *c*) $(\ 1)^5$ *d*) $(\ 1)^{\ 4}$ *e*) 10^2 *f*) $(\ 5)^1$

g) $(\ 10)^2$ *h*) $(\ 10)^3$ *i*) $(\ 10)^{\ 2}$ *j*) $(\ 10)^{\ 3}$ *k*) $10^{\ 2}$ *l*) 7^0

m) 0^2 *n*) $0^{\ 3}$ *o*) $(\ 2)^2$ *p*) $(\ 2)^{\ 3}$ *r*) $5^{\ 2}$ *s*) $(\ 10)^0$

Example:

$$i) \ (\ 10)^{\ 2} \quad \frac{1}{(\ 10)^2} \quad \frac{1}{(\ 10) \ (\ 10)} \quad \frac{1}{100}$$

5. Use the ^ operator to express the following powers (see 24.4):

a) 3^2 *b*) $2^{\ 1}$ *c*) $(\ 2)^3$ *d*) $(\ 5)^{\ 4}$ *e*) $10^{\ 2}$ *f*) $(\ 10)^3$

6. Convert the ^ notation to the regular exponential notation (see 24.4):

a) 5^2 *b*) 5^(2) *c*) (5)^2 *d*) (5)^(2) *e*) (10)^(1)

7. Use the expanded notation to find the value of the following expressions. One case is solved for you as an example.

a) 3^2 *b*) 10^(2) *c*) (5)^2 *d*) (10)^(3) *e*) (2)^(3)

Example:

$$e) \ (\ 2)\wedge(\ 3) \quad (\ 2)^{\ 3} \quad \frac{1}{(\ 2)^3} \quad \frac{1}{(\ 2) \ (\ 2) \ (\ 2)} \quad \frac{1}{8}$$

8. Find the value of each expression containing powers and absolute value functions. One case is solved for you as an example.

a) $| \ 2|^{\ 2}$ *b*) $| \ 10|^{\ 1}$ *c*) $10^{| \ 3|}$ *d*) $| \ 5|^{| \ 2|}$ *e*) $| \ 2|^{| \ 2|}$

Example:

$$e) \ | \ 2|^{| \ 2|} \quad 2^{\ 2} \quad \frac{1}{2^2} \quad \frac{1}{4}$$

25. Exponent Rules

25.1 *(Adding Exponents)* The *product* of two or more powers having a *common base* is a power with the *same base* and an exponent equal to the *sum of exponents*:

$$a^m \quad a^n \quad a^{m \quad n}$$

Example:

$$(\ 2)^3 \quad (\ 2)^5 \quad (\ 2)^{3 \quad 5} \quad (\ 2)^8$$

25.2 *(Subtracting Exponents)* The *division* of two powers having a *common base* is a power with the *same base* and an exponent equal to the *difference* of exponents:

$$\frac{a^m}{a^n} \quad a^{m \quad n}$$

Example:

$$\frac{(\ 3)^5}{(\ 3)^3} \quad (\ 3)^{5 \quad 3} \quad (\ 3)^2$$

25.3 *(Multiplying Exponents)* A *power raised to another power* is a power having the base of the first power and the exponent equal to the *product* of exponents:

$$(a^m)^n \quad a^{m \quad n}$$

Example:

$$((\ 5)^2)^3 \quad (\ 5)^{2 \quad 3} \quad (\ 5)^6$$

25.4 *(Multiplying Bases) Multiplying* two or more numbers and then raising the product to a power is equivalent to raising each number to that power and then multiplying all those powers:

$$(a \quad b \quad ... \quad x)^n \quad a^n \quad b^n \quad ... \quad x^n$$

Example:

$$[(\ 2) \quad (\ 3)]^4 \quad (\ 2)^4 \quad (\ 3)^4$$

25.5 *(Dividing Bases) Dividing* two numbers and then raising the ratio to a power is equivalent to raising each number to that power and then dividing the powers:

$$\frac{a}{b}^{\ n} \quad \frac{a^n}{b^n}$$

Example:

$$\frac{2}{3}^{\ 5} \quad \frac{(\ 2)^5}{(\ 3)^5}$$

25.6 *(Simplification)* Use exponent rules to simplify an expression containing powers.
Example:

$$\frac{[(\ 2)^3 \quad (\ 3)^2]^{\ 2}}{6^{\ 4}} \quad \frac{(\ 2)^{3\ (\ 2)} \quad (\ 3)^{2\ (\ 2)}}{[(\ 2) \quad (\ 3)]^{\ 4}}$$

$$\frac{(\ 2)^{\ 6}}{(\ 2)^{\ 4}} \quad \frac{(\ 3)^{\ 4}}{(\ 3)^{\ 4}} \quad (\ 2)^{\ 6\ (\ 4)} \quad (\ 2)^{\ 2}$$

1. Simplify the following expressions by adding the exponents (see 25.1):

a) $2^3 \quad 2^2$ *b)* $(\ 3)^2 \quad (\ 3)^5$ *c)* $(\ 10)^1 \quad (\ 10)^3$ *d)* $5^5 \quad 5$ *e)* $0^2 \quad 0^3$

2. Rewrite the following expressions as a product of two powers:

a) $4^{3 \quad 2}$ *b)* $(\ 2)^{1 \quad 2}$ *c)* $(\ 10)^{2 \quad 5}$ *d)* $4^{3 \quad 1}$ *e)* $0^{2 \quad 2}$ *f)* $(\ 1)^{3 \quad 1}$

3. Simplify the following expressions by subtracting the exponents (see 25.2):

a) $\frac{2^5}{2^2}$ *b)* $\frac{3^4}{3^1}$ *c)* $\frac{(\ 10)^5}{(\ 10)^3}$ *d)* $\frac{(\ 3)^4}{(\ 3)^4}$ *e)* $\frac{(\ 1)^3}{(\ 1)}$ *f)* $\frac{(\ 4)}{(\ 4)^5}$ *g)* $\frac{3^2}{3^2}$

4. Rewrite the following expressions as a division of two powers:

a) $5^{2 \quad 3}$ *b)* $(\ 5)^{1 \quad 4}$ *c)* $(\ 10)^{\ 3 \quad 6}$ *d)* $1^{3 \quad 5}$ *e)* $4^{1 \quad 2}$ *f)* $(\ 1)^{2 \quad 1}$

5. Simplify (write the expression as one single power as explained at 25.3):

a) $(2^3)^2$ *b)* $(2^{\ 3})^2$ *c)* $(2^{\ 3})^{\ 2}$ *d)* $[(\ 2)^3]^2$ *e)* $(3^{\ 1})^{\ 1}$ *f)* $((\ 7)^{\ 1})^{\ 2}$

6. Expand the brackets (write the expression as a product of two powers as explained at 25.4):

a) $(2 \quad 3)^4$ *b)* $(\ 4 \quad 1)^3$ *c)* $[(\ 5) \quad (\ 2)]^2$ *d)* $[2 \quad (\ 3)]^3$ *e)* $(\ 2 \quad 0)^2$

7. Simplify (write each expression as a single power):

a) $2^2 \quad 2^2$ *b)* $(\ 1)^3 \quad (\ 1)^4$ *c)* $(\ 10)^2 \quad (\ 10)^5$ *d)* $0^1 \quad 0$ *e)* $(\ 5) \quad (\ 5)^2$

8. Expand the brackets (write the expression as a ratio of two powers, as explained at 25.5):

a) $\frac{2}{3}^{\ 4}$ *b)* $\frac{\ 2}{3}^{\ 2}$ *c)* $\frac{\ 1}{2}^{\ 5}$ *d)* $\frac{0}{3}^{\ 3}$ *e)* $\frac{\ 2}{2}^{\ 2}$ *f)* $\frac{\ 5}{3}^{\ 3}$

9. Simplify (write each expression as a single power):

a) $\frac{2^3}{4^3}$ *b)* $\frac{(\ 2)^5}{4^5}$ *c)* $\frac{(\ 1)^2}{(\ 4)^2}$ *d)* $\frac{(\ 6)^7}{(\ 2)^7}$ *e)* $\frac{0^4}{(\ 1)^4}$ *f)* $\frac{2^5}{(\ 3)^5}$

10. Use exponents rules to simplify, then evaluate:

a) $\frac{2^2 \quad 2^3}{2^4}$ *b)* $\frac{(\ 3)^2 \quad (\ 3)^3}{(\ 3) \quad (\ 3)^2}$ *c)* $(\ 10)^2 \quad (\ 10)^{\ 3} \quad (\ 10)^5$

d) $\frac{5^4}{5^3 \quad 5^2}$ *e)* $\frac{(\ 5)^2}{(\ 5)^5 \quad (\ 5)^{\ 2}}$ *f)* $\frac{(\ 10)^7}{(\ 2)^5 \quad (\ 5)^5}$

g) $\frac{[(\ 4)^2 \quad (\ 3)^2]^{\ 2}}{[(\ 6)^{\ 2} \quad (\ 2)^{\ 2}]^3}$ *h)* $\frac{(\ 2)^3 \quad (\ 2)^2}{(\ 2)^4}^{\ 2}$ *i)* $\frac{(\ 4)^3 \quad (\ 8)^2}{16^2 \quad (\ 2)^3}$

j) $\frac{(\ 2) \quad (\ 2)^{\ 2} \quad (\ 2)^3}{(\ 2)^4 \quad ((\ 2)^{\ 2})^3} \quad \frac{(\ 2)^3 \quad (\ 2)^2}{(\ 2)^2 \quad (\ 2)^{\ 1}}^{\ 2}$ *k)* $\frac{[(\ 4)^4 \quad (\ 4)^2]^{\ 3}}{[(\ 8)^{\ 2} \quad (\ 2)^{\ 2}]^2} \quad \frac{2^2 \quad (\ 2)^2}{((\ 4)^{\ 2})^{\ 1}}$

26. Order of Operations (IV)

26.1 *(Negation versus Power)* The operation of raising a number to a power *has a higher priority* than the negation operation (changing a number to its opposite). Examples:

$$2^4 \quad (2^4) \quad 16$$
$$(\ 2)^4 \quad 16$$

26.2 *(Right to Left Order for Exponents)* If an expression contains *consecutive raising operations* to a power, perform the raising operations from *right to left*:

$$a^{b^c} \quad a^{(b^c)} \quad but \quad (a^b)^c \quad a^{b\ c}$$

Examples:

$$2^{3^2} \quad 2^{(3^2)} \quad 2^9 \quad 512$$
$$(2^3)^2 \quad 2^{3\ 2} \quad 2^6 \quad 64$$

26.3 *(Bases or exponents as expressions)* If the base or the exponent of a power is an expression itself, first evaluate the base and the exponent separately, then evaluate the power:

Example: $a^b \quad (a)^\wedge(b)$

$$(\ 2\ \ 3\ \ 4)^{\ 2\ \ 3\ \ 2} \quad (\ 6\ \ 4)^{\ 2\ \ 6} \quad (\ 2)^4 \quad 16$$

26.4 *(Default Order of Operations Updated)* If an expression contains additions, subtractions, multiplications, divisions and powers but *not grouping symbols*, then the default order of operations is:
a) do all the powers operations from right to left
b) do all the multiplication and division operations in the order they appear (from left to right)
c) do all the addition and subtraction operations in the order they appear (from left to right). Example:

$$2\ \ 3^2\ \ 8\ \ 2^2\ \ (\ 3)\ \ (\ 3)^3\ \ 3^2$$
$$2\ \ 9\ \ 8\ \ 4\ \ (\ 3)\ \ (\ 27)\ \ 9$$
$$18\ \ 2\ \ (\ 3)\ \ (\ 3)\ \ 18\ \ 6\ \ 3\ \ 21$$

26.5 *(Order of Operations Algorithm)* If an expression contains *grouping symbols*, use the Order of Operations Algorithm (17.3) and the Default Order of Operations (26.4). Example:

$$(\ 2^3\ \ (\ 4\ \ 8\ \ 2^2)\ \ (\ 4)^2\ \ 1)^{(3\ \ 5)\ (\ 1)}$$
$$(\ 8\ \ (\ 4\ \ 8\ \ 4)\ \ 16\ \ 1)^{(\ 2)\ (\ 1)}$$
$$(\ 8\ \ (\ 4\ \ 2)\ \ 16\ \ 1)^2$$
$$(\ 8\ \ (\ 2)\ \ 16\ \ 1)^2 \quad (16\ \ 16\ \ 1)^2$$
$$(1\ \ 1)^2 \quad 4$$

1. Evaluate the following expressions (see 16.1):

a) 2^2 *b)* $(\ 2)^2$ *c)* $2^{\ 4}$ *d)* $(\ 2)^{\ 4}$ *e)* 10^0 *f)* $(\ 10)^0$

g) $(2)^{\ 1}$ *h)* $(\ 2)^{\ 2}$ *i)* $(\ 123)^0$ *j)* $(\ 2)^{\ 1}$ *k)* 10^1 *l)* $10^{\ 1}$

2. Evaluate the following expressions (see 16.2):

a) 2^{3^2} *b)* $(2^3)^2$ *c)* $2^{(3^2)}$ *d)* 2^{3^2} *e)* $(\ 2)^{3^2}$ *f)* $2^{\ 3^2}$

g) $2^{(\ 3)^2}$ *h)* $(\ 2)^{\ 3^2}$ *i)* $(\ 2)^{(\ 3)^2}$ *j)* $2^{2^{2^2}}$ *k)* $(2^2)^{2^2}$ *l)* $(2^{2^2})^2$

3. Evaluate the following expressions (see 16.3):

a) $(1\ \ 2\ \ 3)^{\ 2\ \ 3}$ *b)* $(\ 8\ \ 2\ \ 2)^{\ 1\ \ 2\ \ 3}$ *c)* $[(1\ \ 3^2)\ \ (\ 2)^2]^{[(2\ \ 3)\ \ 4]\ \ 2}$

d) $(2^{\ 2})^{2\ \ 3\ \ 4}$ *e)* ${\frac{2\ \ 3\ \ 4}{1\ \ 2\ \ 2}}^{\frac{1\ \ 3\ \ 3}{2^2}\ \ 2}$ *f)* $\frac{(\ 10\ \ 5)^{\ 1\ \ 2}\ \ (8\ \ 4)^{\ 9\ \ 3}}{(\ 4\ \ 2)^{\ 2^2}}$

4. Use the default order of operations (see 16.4) to evaluate the following expressions:

a) $1\ \ 2^2\ \ 3^2$ *b)* $2^2\ \ 3^3\ \ 4^2\ \ 2^3\ \ 3^2$ *c)* $(\ 2)^2\ \ (\ 3)^2\ \ (\ 6)^2$

d) $1\ \ 2^2\ \ 3^2\ \ 3$ *e)* $2^2\ \ 4^2\ \ 16\ \ 4^2\ \ 2^4$ *f)* $(\ 3)^3\ \ 3^2\ \ (\ 2)^3\ \ (\ 5)\ \ 2^2$

5. Use the order of operations algorithm (see 16.4) to evaluate the following expressions:

a) $2\ \ 2\ \ 2\ \ 2\ \ 2^2$ *b)* $2^2\ \ 2\ \ 2\ \ 2\ \ 2$ *c)* $2\ \ 2^2\ \ 2\ \ 2\ \ 2$

d) $2\ \ 2^2\ \ 2\ \ 2\ \ 2$ *e)* $2\ \ 2\ \ 2\ \ 2^2\ \ 2$ *f)* $2^2\ \ 2\ \ 2\ \ 2\ \ 2$

g) $2\ \ 2\ \ (2\ \ 2\ \ 2)^2$ *h)* $2^{2\ \ 2\ \ 2}\ \ 2\ \ 2$ *i)* $2\ \ 2^{2\ \ 2\ \ 2}\ \ 2$

j) $2\ \ 2^{2\ \ 2\ \ 2}\ \ 2$ *k)* $2\ \ (2\ \ 2\ \ 2)^2\ \ 2$ *l)* $(2^2\ \ 2)\ \ 2\ \ 2\ \ 2$

6. Use the order of operations algorithm (see 16.4) to evaluate the following expressions:

a) $(1\ \ 3\ \ 4\ \ 3^2)^3\ \ 5$ *b)* $(3\ \ 2^2\ \ 5)\ \ (\ 10\ \ 5\ \ 2^2)\ \ (\ 3^2\ \ 2\ \ 3^2\ \ 2^3)^2$

c) $2^{(\ 3\ \ 2)\ (\ 3)}\ \ (\ 3\ \ 2\ \ 4)^2$ *d)* $[\ 1\ \ (\ 3\ \ 5)^2\ \ (\ 1\ \ 2)]^2\ \ [(2^3)\ \ (2^2)\ \ 1]^3\ \ 3$

e) ${\frac{2^4\ \ (2^2\ \ 1)\ \ (2^2\ \ 1)}{2^3\ \ (\ 4)\ \ 2^2\ \ 1}}^2$ *f)* $[(2\ \ 2^2)^{(2\ \ 4)^{(2\ \ 3)}}]^{(2\ \ 2^2)}$

g) $\frac{2\ \ (\ 2)^{\ 2}\ \ (\ 2)^3\ \ 2^3}{2^2\ \ 2^6\ \ (\ 2)^2\ \ (\ 2)^{\ 3}}$ *h)* ${\frac{3^2\ \ 2^2\ \ 1}{2^2\ \ 2^3\ \ 2}}^3\ \ {\frac{(\ 3\ \ 4)\ \ (2^2\ \ 2\ \ 3)\ \ 1}{(3^2\ \ 3)\ \ 2\ \ 3}}^2$

i) $(2^{\ 3})^{\ 2}\ \ 2^{2^2}\ \ (3^2\ \ 2^3)^{3^2}\ \ (5^2\ \ 2^5\ \ 2\ \ 5)^2\ \ (\ 3)^2\ \ (\ 3^2\ \ 2^3\ \ 2^4)^2\ \ 2^5$

j) ${\frac{3^2}{2^2\ \ 1}}^{{\frac{4^2\ \ 2\ \ 5}{1\ \ 2}}^2\ \ 2}\ \ {\frac{\frac{2^4\ \ (\ 2)}{(\ 2)^2}}{\frac{3^2\ \ 5}{1^0\ \ 2^0}}}^3\ \ \frac{(\ 3)^2\ \ 2^2\ \ 1}{4}\ \ {\frac{2^4\ \ 2}{1\ \ (3^2\ \ 1)}}^2\ \ 2$

27. Divisors

27.1 *(Divisors)* An integer *a* is a *divisor* of the integer *b* if dividing *b* by *a* leaves *no remainder*. In this case there is an integer *c* so:

$$\frac{b}{a} \quad c \qquad b \quad a \quad c$$

Example: *-3* is a divisor of *12* because:

$$\frac{12}{3} \quad 4 \qquad 12 \quad (\ 3) \quad (\ 4)$$

27.2 *(Definitions)* Both *a* and *c* are called *divisors* or *factors* of *a*.
Both *a* and *c divide a*.
b is called a *multiplier* of both *a* and *c*.
b is *divisible* by both *a* and *c*.
For the example above:
Both *-3* and *-4* are divisors of *12*
Both *-3* and *-4* divide *12*
12 is divisible by both *-3* and *-4*

27.3 *(Improper Divisors)* Given any integer *a* (except *0* and *1*) there are at least 4 divisors of this number that are improper divisors: *1*, *-1*, *a*, *-a*. Indeed:

$$\frac{a}{1} \quad a; \frac{a}{1} \quad a; \frac{a}{a} \quad 1; \frac{a}{a} \quad 1$$

Example: The improper divisors of *-5* are: *1*, *-1*, *-5*, *+5*.

27.4 *(Proper Divisors)* Any divisor of an integer that *is not an improper divisor* is called a *proper divisor*.
Example: The proper divisors of *-6* are: *+2*, *-2*, *+3*, and *-3*

27.5 *(Prime Numbers)* A *prime number* is a positive integer (a natural number) with no proper divisors. The only divisors of a prime number are *1* and the *number itself* (if you ignore *-1* and the opposite of that number).

Example: *7* is a prime number. The only natural numbers that are divisors of *7* are the improper divisors *1* and *7*.

27.6 *(Composite Numbers)* A natural number that is not a prime number is called composite number. A composite number has proper divisors (different from *1* and the *number itself*).
Example: *12* is composite number. The improper divisors are *1* and *12*. The proper divisors are *2*, *3*, *4*, and *6*. For this example the negative divisors were ignored.

27.7 *(Number 1)* Conventionally, number *1* is neither a prime nor a composite number.

1. Find the logical value (true or false) of each statement:

a) 5 *is a divisor of* 20 *b*) 3 *divides* 10

c) 16 *is a multiple of* 4 *d*) 24 *is divisible by* 2

e) 3 *is a divisor of* 8 *f*) 5 *divides* 20

g) 14 *is a multiple of* 6 *h*) 2 *divides* 11

2. Find the improper divisors of each integer:

a) 10 *b*) 8 *c*) 5 *d*) 20 *e*) 11 *f*) 3

3. Find the proper divisors of each integer:

a) 12 *b*) 6 *c*) 15 *d*) 10 *e*) 6 *f*) 16

4. Find all divisors of each integer:

a) 9 *b*) 18 *c*) 24 *d*) 20 *e*) 30 *f*) 21

5. Find all prime numbers between 1 and 30.

6. Find all composite numbers between 31 and 50.

7. Classify each of the following numbers as either prime, composite or neither.

a) 11 *b*) 21 *c*) 31 *d*) 41 *e*) 51 *f*) 53 *g*) 57 *h*) 1

8. Use the Eratosthenes Sieve algorithm (below) to find the prime numbers between 1 and 100:

1. Create the list of all whole numbers between 2 and 100: ***list A*** *(see the list below).*
2. Create an empty list and name it: ***list B*** *(this is the list of prime numbers).*
3. Move the first number from list A to list B, then remove from list A all that number's multiples.
4. Repeat step 3 until no more numbers are left in list A.

The list of prime numbers is the list B.

List A:

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

List B:

28. Divisibility Rules

28.1 *(Divisibility by 2)* An integer is divisible by *2* if the last digit is *2*, *4*, *6*, *8*, or *0*. A number divisible by *2* is called an *even number*. Examples: *24*, *-12*, *-30*.

28.2 *(Divisibility by 3)* An integer is divisible by *3* if the sum of all digits is divisible by *3*. Examples: *-111*, *102*, *750*.

28.3 *(Divisibility by 4)* An integer is divisible by *4* if the number defined by the last two digits is divisible by 4. Examples: *1020*, *-132*, *-144*..

28.4 *(Divisibility by 5)* An integer is divisible by *5* if the last digit is *5* or *0*.

28.5 *(Divisibility by 6)* An integer is divisible by *6* if the number is divisible by *2* and *3*. Examples: *102*, *-222*, *1014*.

28.6 *(Divisibility by 7)* Use the following algorithm:
a) Drop the last digit and double it
b) Subtract the result from the remaining number.
c) If the final result is divisible by *7* then the original number is also divisible by *7*.
Example: 791; 79 2 1 77

28.7 *(Divisibility by 8)* An integer is divisible by *8* if the number defined by the last three digits is divisible by *8*. Examples: *816*, *-8192*, *-2000*.

28.8 *(Divisibility by 9)* An integer is divisible by *9* if the sum of all digits is divisible by *9*. Examples: *369*, *-252*, *126*.

28.9 *(Divisibility by 10)* An integer is divisible by *10* if the last digit is *0*.

28.10 *(Divisibility by 11)* Alternately add and subtract the digits from left to right. If the number you get is divisible by 11 then the original number also is. Example: *1353*.

$$1 \quad 3 \quad 5 \quad 3 \quad 0; \quad \frac{0}{11} \quad 0$$

28.11 *(Divisibility by 12)* An integer is divisible by *12* if the number is divisible by *3* and *4*. Examples: *22344*, *-132*, *-8652*.

28.12 *(Divisibility by 13)* Use the following algorithm:
a) Drop the last digit and multiply it by 9
b) Subtract the result from the remaining number.
c) If the final result is divisible by 13 then the original number also is. Example: *3003*.

$$300 \quad 9 \quad 3 \quad 273$$

$$27 \quad 9 \quad 3 \quad 0; \quad \frac{0}{13} \quad 0$$

1. For each of the following numbers, find whether or not they are divisible by 2:

a) 55 *b*) 14 *c*) 0 *d*) 21 *e*) 112 *f*) 31 *g*) 36

2. For each of the following numbers, find whether or not they are divisible by 3:

a) 1234 *b*) 222 *c*) 1111 *d*) 201 *e*) 765 *f*) 130 *g*) 12345

3. For each of the following numbers, find whether or not they are divisible by 4:

a) 120 *b*) 312 *c*) 510 *d*) 701 *e*) 966 *f*) 148 *g*) 12386

4. For each of the following numbers, find whether or not they are divisible by 5:

a) 220 *b*) 115 *c*) 1225 *d*) 700 *e*) 465 *f*) 444 *g*) 12777

5. For each of the following numbers, find whether or not they are divisible by 6:

a) 246 *b*) 135 *c*) 1158 *d*) 900 *e*) 465 *f*) 888 *g*) 9654

6. For each of the following numbers, find whether or not they are divisible by 7:

a) 231 *b*) 635 *c*) 1152 *d*) 861 *e*) 465 *f*) 885 *g*) 23345

7. For each of the following numbers, find whether or not they are divisible by 8:

a) 248 *b*) 620 *c*) 1152 *d*) 556 *e*) 890 *f*) 896 *g*) 9870

8. For each of the following numbers, find whether or not they are divisible by 9:

a) 1234 *b*) 6588 *c*) 108 *d*) 777 *e*) 981 *f*) 666 *g*) 5553

9. For each of the following numbers, find whether or not they are divisible by 10:

a) 1230 *b*) 5555 *c*) 1000 *d*) 7890 *e*) 999

10. For each of the following numbers, find whether or not they are divisible by 11:

a) 1232 *b*) 6171 *c*) 308 *d*) 1111 *e*) 22 *f*) 642 *g*) 23432

11. For each of the following numbers, find whether or not they are divisible by 12:

a) 240 *b*) 600 *c*) 324 *d*) 222 *e*) 111 *f*) 12345 *g*) 66

12. For each of the following numbers, find whether or not they are divisible by 13:

a) 130 *b*) 3900 *c*) 182 *d*) 1313 *e*) 11 *f*) 100 *g*) 74191

13. Use a calculator and the algorithms in 28.1 to 28.12 to find if:

a) 3 *is a divisor of* 12345678987654321
b) 999888777666555444 *is a multiple of* 11
c) 6 *is a divisor of* 135798642012345
d) 122333444455555666666 *is a multiple of* 12
d) 9 *divide* 98765432100123456789

29. Prime Factorization

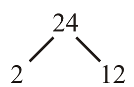
29.1 *(Fundamental Theorem of Arithmetic)* Any natural number except *1* can be written as a product of prime numbers in a unique way called the *prime factorization* of that number. Use exponents to make the expression simpler. Example:

$$1500 \quad 2 \quad 2 \quad 3 \quad 5 \quad 5 \quad 5 \quad 2^2 \quad 3^1 \quad 5^3$$

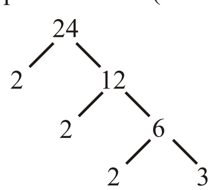
29.2 *(Factor Tree Method)* The factor tree method is a method to find the prime factorization of a natural number. The *root* of the tree is the number itself. For example:

24

Using divisibility rules try to find a divisor of the given number. In our case *2* is a divisor. By dividing *24* by *2* you get *12*. So:



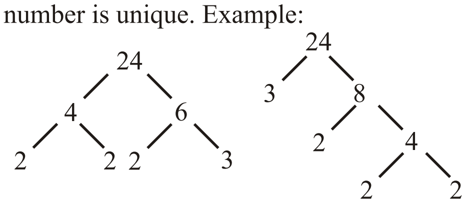
2 is a prime factor and so is a leaf of the tree. Let's continue the process with *12* until we get only prime factors (*leaves*):



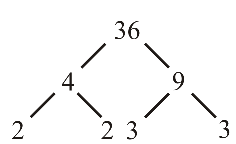
All leaves of the tree generate the prime factorization of the given number:

$$24 \quad 2 \quad 2 \quad 2 \quad 3 \quad 2^3 \quad 3^1$$

29.3 *(Uniqueness)* Although for a natural number more than one factor tree is possible, the prime factorization of that number is unique. Example:



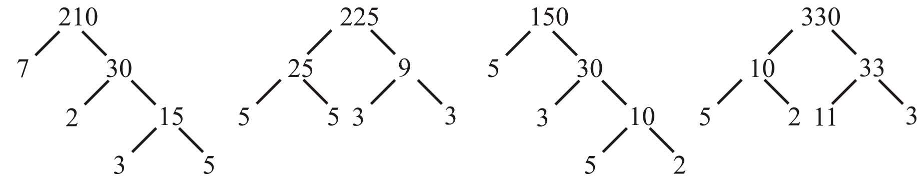
29.4 *(Integers)* Prime factorization can be extended to integers (except *0*, *1*, and *-1*) if only positive factors are considered. Example: *-36*



So:

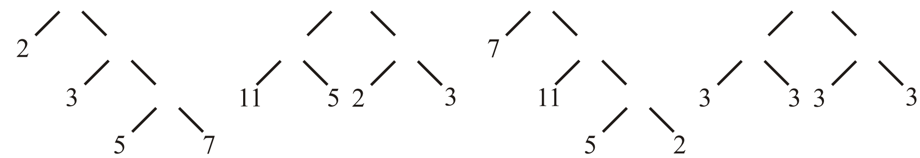
$$36 \quad 2^2 \quad 3^2$$

1. Use the following factor trees and write down for each case the prime factorization of the number placed at the root of the tree:



A) B) C) D)

2. Complete the following factor trees and write down the prime factorization:



A) B) C) D)

3. Build a factor tree for each number and write down its prime factorization:

a) 40 *b*) 64 *c*) 80 *d*) 100 *e*) 250 *f*) 1024 *g*) 350 *h*) 83

4. Build at least 4 different factor trees for the number 600.

5. Build a factor tree for each of the following negative integers and write down its prime factorization:

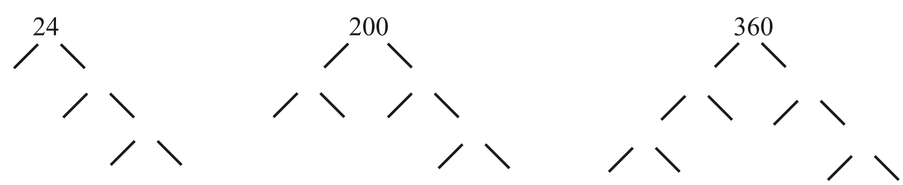
a) 12 *b*) 88 *c*) 300 *d*) 4096 *e*) 17 *f*) 12345

6. Find the value of each number given by its prime factorization and then build a factor tree:

a) $2^3 \quad 3^2$ *b*) $3^1 \quad 5^2 \quad 7^1$ *c*) $2^4 \quad 5^2$ *d*) $2^2 \quad 3^2 \quad 5^2$

e) 2^8 *f*) 3^5 *g*) $2^5 \quad 5^2$ *h*) $2^1 \quad 3^1 \quad 5^1 \quad 7^1 \quad 11^1$

7. Complete the following factor trees and write down the prime factorization:



A) B) C)

8. Three numbers a, b, and c are given by their prime factorization:

$a \quad 2^3 \quad 3^2 \quad 5^2 \qquad b \quad 2^1 \quad 3^4 \quad 5^3 \quad 7^2 \qquad c \quad 2^2 \quad 3^1 \quad 5^3 \quad 7^1$

Use the exponents rules to find the prime factorization for:

a) $a \quad b$ *b*) bc/a *c*) a^2 *d*) $\frac{a \quad b}{c}$ *e*) $a \quad b \quad c$ *f*) $\frac{a \quad b^2}{c^2}$

30. Set of Divisors

30.1 *(Set)* A set is a *collection* of elements. One way to define a set is to enumerate all the elements that belong to the set..
Example: {1,2,5}

Note that the elements are separated by commas and written inside braces {}. This is called the *set notation*.

30.2 *(Multiplying Sets)* The product of two sets is a set of products of each element of the first set by each element of the second set. Example:

{1,2,3} {5,7}

{1 5,1 7,2 5,2 7,3 5,3 7}

{5,7,10,14,15,21}

30.3 *(The Set of Divisors)* To find the set of all divisors of a natural number:
a) find the prime factorization
b) for each prime factor, build a set having elements the prime factor raised to all possible powers starting 0 and ending at the maximum power as found in the prime factorization
c) multiply the sets you get in step b)
Example:

90 2^1 3^2 5^1

2^1 {1,2}

3^2 {1,3,9}

5^1 {1,5}

{1,2} {1,3,9} {1,5}

{1 1,1 3,1 9,2 1,2 3,2 9} {1,5}

{1,3,9,2,6,18,} {1,5}

{1 1,1 5,3 1,3 5,9 1,9 5,2 1,2 5,6 1,6 5,18 1,18 5}

{1,5,3,15,9,45,2,10,6,30,18,90}

30.4 *(Integral Divisors)* Given an integer you can use the algorithm presented above to find all integral divisors of that number.
Example:

24 2^3 3^1

2^3 {1,2,4,8}

3^1 {1,3}

{1,2,4,8} {1,3}

{1 1,1 3,2 1,2 3,4 1,4 3,8 1,8 3}

{1,3,2,6,4,12,8,24}

So, the set of all integral divisors is:

{ 1, 3, 2, 6, 4, 12, 8, 24}

1. Use the set notation to express the following sets:

A) the set of prime numbers less than 10
B) the set of divisors of 10
C) the set of first 5 natural numbers starting with 0
D) the set of first positive 5 multiples of 3
E) the set of odd numbers between 10 and 20

2. Use the rule of "multiplying sets" presented in 30.2 and find the "product" of the following sets:

a) {1,2} {1,2} *b)* {1,2,3} {2,3} *c)* {2,5} {3,4} *d)* {1,2} {1,2} {1,2}
e) {1} {5} *f)* {1,2} {3,4} *g)* { 1,2} {1, 2} *h)* {1} {1,2} {1,2,3}

3. For each of the following powers, build the set of all its divisors (see 30.3):

a) 2^1 *b)* 2^2 *c)* 2^3 *d)* 2^4
e) 3^2 *f)* 3^3 *g)* 2^{10} *h)* 5^3

Example:

d) 2^4 $\{2^0,2^1,2^2,2^3,2^4\}$ {1,2,4,8,16}

4. For each of the following natural numbers, find the set of positive divisors (see 30.3):

a) 6 *b)* 10 *c)* 16 *d)* 20 *e)* 36 *f)* 100 *g)* 625 *h)* 2048

5. For each of the following integers, find the set of all divisors (positive and negative) (see 30.3):

a) 8 *b)* 40 *c)* 160 *d)* 200 *e)* 360 *f)* 1000 *g)* 512

6. For each natural number, find the number of divisors (see the example below):

a) 12 *b)* 40 *c)* 64 *d)* 200 *e)* 640 *f)* 6000 *g)* 2500 *h)* 5120

Example:

f) 6000

Perform the prime factorization: 6000 2^4 3^1 5^3

Add 1 to all exponents and multiply them: (4 1) (1 1) (3 1) 5 2 4 40

This is the number of all positive divisors.

7. Compute the following operations with sets of numbers (see 30.2 for multiplication). One case is solved for you as an example:

a) {1,2} {1,2} *b)* {1,2,3} {2,3} *c)* $\{2,5\}^2$ *d)* $\{1,2\}^3$ *e)* {2,4,6} {1,2}

Example:

c) $\{2,5\}^2$ {2,5} {2,5} {2 2,2 5,5 2,5 5} {4,10,10,25} {4,10,25}

Note: In a set any element is unique (appears only once).

31. Highest Common Factor (HCF)

31.1 *(HCF)* The *Highest Common Factor* (HCF) of two or more natural numbers is the highest common natural number into which you can divide these numbers evenly (without a remainder). Example:

$Given: 8, 12, and\ 20$

$HCF \quad 4$

$\frac{8}{4} \quad 2; \frac{12}{4} \quad 3; \frac{20}{4} \quad 5$

31.2 *(The Algorithm for finding HCF)* The following algorithm allows you to find the *HCF* for any set of natural numbers:
a) perform the prime factorization of each number
b) complete the missing prime factors with prime factors raised to power *0*
c) HCF is the number with all the prime factors raised to the *lowest exponents*.
Example:

$Given: 84, 120, and\ 180$

$a)\ 84 \quad 2^2 \quad 3^1 \quad 7^1; 120 \quad 2^3 \quad 3^1 \quad 5^1;$

$180 \quad 2^2 \quad 3^2 \quad 5^1$

$b)\ 84 \quad 2^2 \quad 3^1 \quad 5^0 \quad 7^1;$

$120 \quad 2^3 \quad 3^1 \quad 5^1 \quad 7^0; 180 \quad 2^2 \quad 3^1 \quad 5^0 \quad 7^1$

$c)\ HCF \quad 2^2 \quad 3^1 \quad 5^0 \quad 7^0 \quad 2^2 \quad 3^1 \quad 12$

31.3 *(HCF for integers)* To calculate the HCF for a set of integers, consider only the positive divisors of these numbers. In this way, the HCF will always be a natural number. Example:

$Given: \quad 200\ and \quad 180$

$a) \quad 200 \quad 2^3 \quad 5^2; \quad 180 \quad 2^2 \quad 3^2 \quad 5^1$

$b) \quad 200 \quad 2^3 \quad 3^0 \quad 5^2; \quad 180 \quad 2^2 \quad 3^2 \quad 5^1$

$c)\ HCF \quad 2^2 \quad 3^0 \quad 5^1 \quad 20$

31.4 *(Factoring)* Given an expression with two or more terms you can always factor the HCF of these terms. Example:

$12 \quad 30 \quad 42 \quad 6\ (\quad 2 \quad 5 \quad 7)$

31.5 *(HCF Function)* The function *HCF* has as *input* a list of natural or integral numbers separated by commas, and as *output* the *HCF* of these numbers. Examples:

$HCF(10,15) \quad 5$

$HCF(12, \quad 30, 60) \quad 6$

1. For each case: find the set of the positive divisors of all the given numbers then choose the highest common divisor (factor):

$a) \quad 12; 20 \quad b) \quad 8; 36; 48 \quad c) \quad 64; 24 \quad d) \quad 81; 36; 90 \quad e) \quad 250; 100 \quad f) \quad 18; 60$

Example:

$f) \quad 18; 60$

$18 \quad 2^1 \quad 3^2 \qquad \{1,2\} \quad \{1,3,9\} \quad \{1,3,9,2,6,18\} \quad \{1,2,3,6,9,18\}$

$60 \quad 2^2 \quad 3^1 \quad 5^1 \qquad \{1,2,4\} \quad \{1,3\} \quad \{1,5\} \quad \{1,3,2,6,4,12\} \quad \{1,5\}$

$\{1,5,3,15,2,10,6,30,4,20,12,60\} \quad \{1,2,3,4,5,6,10,12,15,20,30,60\}$

$HCF \quad 6$

2. For each case, find the HCF by factoring out the common factors one by one (one case is solved for you as an example):

$a) \quad 8; 12 \quad b) \quad 6; 20; 36 \quad c) \quad 24; 32 \quad d) \quad 35; 45; 20 \quad e) \quad 40; 48 \quad f) \quad 70; 105$

Example:

$f) \quad \{70;105\} \quad 5 \quad \{14;21\} \quad 5 \quad 7 \quad \{2;3\}$

The HCF is $5 \quad 7 \quad 35$

3. For each case use the algorithm presented in 31.2 to find the HCF:

$a) \quad 24; 30 \quad b) \quad 18; 60; 72 \quad c) \quad 128; 81 \quad d) \quad 64; 40; 100 \quad e) \quad 225; 300$

4. For each case use the algorithm presented in 31.2 to find the HCF (see also 31.3):

$a) \quad 400; \quad 250 \quad b) \quad 30; \quad 36; \quad 60 \quad c) \quad 21; 40 \quad d) \quad 125; \quad 225; \quad 625$

5. Rewrite the following expressions by factoring out the HCF:

$a) \quad 24 \quad 36 \qquad b) \quad 64 \quad 80 \qquad c) \quad 125 \quad 225 \qquad d) \quad 240 \quad 100 \quad 350$

$e) \quad 12 \quad 30 \quad 18 \quad f) \quad 200 \quad 250 \quad 500 \quad g) \quad 64 \quad 256 \quad h) \quad 60 \quad 72 \quad 90$

6. Simplify the following expressions by canceling out one by one the common factors (one case is solved for you as an example):

$a) \ \frac{9 \quad 6}{15 \quad 12} \qquad b) \ \frac{15 \quad 30}{20 \quad 25} \qquad c) \ \frac{30 \quad 18 \quad 36}{42 \quad 54} \qquad d) \ \frac{40 \quad 24 \quad 64}{80 \quad 96}$

$e) \ \frac{30 \quad 60 \quad 80}{40 \quad 90} \qquad f) \ \frac{60 \quad 72 \quad 120}{36 \quad 108} \qquad g) \ \frac{75 \quad 50}{125 \quad 150} \qquad h) \ \frac{17 \quad 51}{34 \quad 102}$

Example:

$c) \ \frac{30 \quad 18 \quad 36}{42 \quad 54} \quad \frac{2 \quad (15 \quad 9 \quad 18)}{2 \quad (\quad 21 \quad 27)} \quad \frac{15 \quad 9 \quad 18}{21 \quad 27} \quad \frac{3 \quad (5 \quad 3 \quad 6)}{3 \quad (\quad 7 \quad 9)} \quad \frac{5 \quad 3 \quad 6}{7 \quad 9} \quad \frac{8}{2} \quad 4$

7. Simplify the following expressions by canceling out the HCF.

$a) \ \frac{14 \quad 35}{21 \quad 42} \qquad b) \ \frac{24 \quad 40}{64 \quad 56} \qquad c) \ \frac{24 \quad 36 \quad 60}{72 \quad 84} \qquad d) \ \frac{45 \quad 60 \quad 90}{150 \quad 115}$

$e) \ \frac{32 \quad 64 \quad 80}{48 \quad 96} \qquad f) \ \frac{50 \quad 75 \quad 125}{100 \quad 150} \qquad g) \ \frac{26 \quad 52}{65 \quad 130} \qquad h) \ \frac{7 \quad 3}{2 \quad 5}$

Example: $f) \ \frac{50 \quad 75 \quad 125}{100 \quad 150} \quad \frac{25 \quad (2 \quad 3 \quad 5)}{25 \quad (\quad 4 \quad 6)} \quad \frac{2 \quad 3 \quad 5}{4 \quad 6} \quad \frac{4}{2} \quad 2$

32. Least Common Multiple (LCM)

32.1 *(LCM)* The *Least Common Multiple* (LCM) of two or more natural numbers is the least common natural number which can be divided evenly (without a remainder) by each of these numbers. Example:

$$Given\ 4\ and\ 6;\quad LCM\quad 12$$

$$\frac{12}{4}\quad 3;\quad \frac{12}{6}\quad 2$$

32.2 *(The Algorithm for finding LCM. Solution 1)* List all the multiples of each given number until you find a first common multiple. That is the *Least Common Multiple*. Example:

$Given\ 3,4,\ and\ 6$

$3\quad 3,6,9,12,15,18,21,24,27,30,...$

$4\quad 4,8,12,16,20,24,28,...$

$6\quad 6,12,18,24,30,...$

$LCM\quad 24$

32.3 *(The Algorithm for finding LCM. Solution 2)* The following algorithm allows you to find the *LCM* for any set of natural numbers:
a) perform the prime factorization of each number
b) complete the missing prime factors with prime factors raised to power *0*
c) LCM is the number having all the prime factors raised to the *greatest* exponents.
Example:

$Given\ 8,12,\ and\ 25;\quad 8\quad 2^3\quad 3^0\quad 5^0$

$12\quad 2^2\quad 3^1\quad 5^0;\quad 25\quad 2^0\quad 3^0\quad 5^2$

$LCM\quad 2^3\quad 3^1\quad 5^2\quad 600$

32.4 *(LCM & HCF)* The product of any two natural numbers (except 1) is equal to the product of LCM and HCF of these numbers. Example:

$Given\ 12\ and\ 15$

$LCM\quad 60;\quad HCF\quad 3$

$12\quad 15\quad 180\quad LCM\quad HCF$

32.5 *(LCM for integers)* To calculate the LCM for a set of integers, consider only the positive multiples of these numbers. In this way, the LCM will always be a natural number. Example:

$Given\quad 6,\ and\quad 15;\quad 6\quad 2^1\quad 3^1\quad 5^0$

$15\quad 2^0\quad 3^1\quad 5^1;\quad LCM\quad 2^1\quad 3^1\quad 5^1\quad 30$

32.6 *(LCM Function)* The function *LCM* has as *input* a list of natural or integral numbers separated by commas, and as *output* the *LCM* of these numbers. Examples:

$LCM(10,15)\quad 30;\ LCM(\ \ 8,\ \ 15)\quad 120$

1. For each set of numbers, list the multiples until you can identify the LCM (see 32.2):

$a)\ 6;15\quad b)\ 3;8;10\quad c)\ 5;12\quad d)\ 4;6;15\quad e)\ 25;35\quad f)\ 12;20$

2. For each set of numbers, use the algorithm presented in 32.3 to find the LCM:

$a)\ 4;10\quad b)\ 10;15;18\quad c)\ 16;20\quad d)\ 5;12;8\quad e)\ 30;25\quad f)\ 24;32$

3. For each pair of numbers, prove that the product of the numbers is equal to the product between the HCF and the LCM of these numbers:

$a)\ 10;12\quad b)\ 10;15\quad c)\ 12;30\quad d)\ 6;8\quad e)\ 10;25\quad f)\ 14;16$

4. For each set of integers find the LCM:

$a)\ 3;8\quad b)\ 20;25;\ 30\quad c)\ 12;\ 16\quad d)\ 18;\ 12;5\quad e)\ 20;\ 25$

5. Find the value of the following expressions containing HCF and LCM functions (see 31.5 and 32.6):

$a)\ HCF(6,8)\qquad b)\ LCM(12,16)$

$c)\ HCF(20,24)\ \ LCM(6,8)\qquad d)\ \frac{HCF(48,32)}{LCM(15,20)}$

$e)\ \frac{25\quad 35}{LCM(25,35)\quad HCF(25,35)}\qquad f)\ HCF(120,100,80)$

$g)\ \frac{HCF(160,100,50)}{LCM(16,10,5)}\qquad h)\ \frac{10\quad 15\quad 25}{LCM(10,15,25)\quad HCF(10,15,25)}$

Example:

$f)\ HCF(120,100,80)$

$120\quad 2^3\quad 3^1\quad 5^1;100\quad 2^2\quad 5^2;80\quad 2^4\quad 5^1$

$HCF(120,100,80)\quad 2^2\quad 5^1\quad 20$

6. LCM is used to add (or subtract) two or more fractions with unlike denominators. The Least Common Denominator (LCD) is the LCM of all denominators. To add the following fractions, convert them into equivalent fractions having a denominator equal to LCD and then add (or subtract) numerators. One case is solved for you as an example:

$a)\ \frac{1}{2}\ \frac{1}{3}\quad b)\ \frac{2}{3}\ \frac{1}{6}\quad c)\ \frac{3}{4}\ \frac{5}{6}\quad d)\ \frac{1}{10}\ \frac{1}{15}\quad e)\ \frac{1}{12}\ \frac{1}{16}\quad f)\ \frac{3}{8}\ \frac{1}{10}$

Example: $c)\ \frac{3}{4}\ \frac{5}{6}\ \frac{3}{3}\ \frac{3}{4}\ \frac{2}{2}\ \frac{5}{6}\ \frac{9}{12}\ \frac{10}{12}\ \frac{9\quad 10}{12}\ \frac{19}{12}$

7. LCM is used to add (or subtract) two or more rational numbers with unlike denominators. The Least Common Denominator (LCD) is the LCM of all denominators. To add the following rational numbers, convert them into equivalent numbers having a denominator equal to LCD and then add (or subtract) numerators. One case is solved for you as an example:

$a)\ \frac{1}{2}\ \frac{1}{3}\quad b)\ \frac{2}{3}\ \frac{5}{6}\quad c)\ \frac{1}{4}\ \frac{5}{6}\quad d)\ \frac{1}{10}\ \frac{1}{15}\quad e)\ \frac{5}{12}\ \frac{1}{16}$

Example: $d)\ \frac{1}{10}\ \frac{1}{15}\ \frac{1}{10}\ \frac{1}{15}\ \frac{3}{3}\ \frac{1}{10}\ \frac{2}{2}\ \frac{1}{15}\ \frac{3}{30}\ \frac{2}{30}\ \frac{3\quad 2}{30}\ \frac{1}{30}$

33. Square Roots

33.1 *(Square Roots)* The square root operation is the *inverse operation* to raising a number to the power 2:

$\sqrt{a}$ *is that x so* $x^2 \quad a$

Example: $\sqrt{4} \quad 2; \; 2^2 \quad 4$

33.2 *(Radicand)* Because any real number (including an integer) squared is positive or 0, the *radicand* (the expressions inside the radical sign) is positive or 0:

$\sqrt{a}$ *is real if* $a \quad 0$

Example:

$\sqrt{\quad 16}$ *is not real because* $16 \quad 0$

33.3 *(Principal Root)* If more numbers squared are equal to the radicand, then by convention the square root is represented by the positive number called the *principal root*:

$\sqrt{a} \quad 0$

Example: $(\;3)^2 \quad 9; \; 3^2 \quad 9; \; \sqrt{9} \quad 3$

33.4 *(Basic Square Roots)* Here is a list of some basic *roots* (square roots):

$0^2 \quad 0 \quad \sqrt{0} \quad 0; \; 1^2 \quad 1 \quad \sqrt{1} \quad 1$

$2^2 \quad 4 \quad \sqrt{4} \quad 2; \; 3^2 \quad 9 \quad \sqrt{9} \quad 3$

$4^2 \quad 16 \quad \sqrt{16} \quad 4; \; 5^2 \quad 25 \quad \sqrt{25} \quad 5;$

33.5 *(Square Root of a Square)* The square root of a *positive number squared* gives you back the original number:

$\sqrt{3^2} \quad \sqrt{9} \quad 3$

33.6 *(Square root of a square)* The square root of a *negative number squared* gives you the opposite of the original number:

$\sqrt{(\;3)^2} \quad \sqrt{9} \quad 3;$ *not* 3

33.7 *(Algorithm for finding the square root)* To find the square root of a number:
a) do the prime factorization
b) the square root has the same factors as the original number, but all exponents are *divided by 2*
Example:

$\sqrt{7056} \quad ?; \; 7056 \quad 2^4 \quad 3^2 \quad 7^2$

$\sqrt{7056} \quad 2^2 \quad 3^1 \quad 7^1 \quad 84$

33.8 *(Real Numbers)* If the radicand of a square root is not a perfect square, then the square root of that number is a *real number*. You can use a calculator to find a decimal approximation of the real number. Example:

$\sqrt{2} \quad 1.141$

1. Find the squares of the following numbers:

a) 0 *b*) 1 *c*) 2 *d*) 3 *e*) 4 *f*) 5 *g*) 6 *h*) 7 *i*) 8
j) 9 *k*) 10 *l*) 11 *m*) 12 *n*) 15 *o*) 20 *p*) 50 *q*) 100 *r*) 200

2. Find the following square roots (use the results you have got at Exercise 1):

a) $\sqrt{1}$ *b*) $\sqrt{4}$ *c*) $\sqrt{0}$ *d*) $\sqrt{25}$ *e*) $\sqrt{81}$ *f*) $\sqrt{36}$
g) $\sqrt{9}$ *h*) $\sqrt{64}$ *i*) $\sqrt{121}$ *j*) $\sqrt{400}$ *k*) $\sqrt{100}$ *l*) $\sqrt{10000}$
m) $\sqrt{2500}$ *n*) $\sqrt{40000}$ *o*) $\sqrt{49}$ *p*) $\sqrt{16}$ *q*) $\sqrt{144}$ *r*) $\sqrt{16}$

3. Find the value of the following expressions containing square roots of squares (see 33.5):

a) $\sqrt{1^2}$ *b*) $\sqrt{3^2}$ *c*) $\sqrt{0^2}$ *d*) $\sqrt{7^2}$ *e*) $\sqrt{5^2}$ *f*) $\sqrt{9^2}$
g) $\sqrt{10^2}$ *h*) $\sqrt{13^2}$ *i*) $\sqrt{20^2}$ *j*) $\sqrt{100^2}$ *k*) $\sqrt{123^2}$ *l*) $\sqrt{99999^2}$

4. Find the value of the following expressions containing square roots of squares (see 33.6):

a) $\sqrt{(\;5)^2}$ *b*) $\sqrt{(\;1)^2}$ *c*) $\sqrt{(\;7)^2}$ *d*) $\sqrt{(\;8)^2}$ *e*) $\sqrt{(\;3)^2}$
f) $\sqrt{(\;11)^2}$ *g*) $\sqrt{(\;10)^2}$ *h*) $\sqrt{(\;30)^2}$ *i*) $\sqrt{(\;100)^2}$ *j*) $\sqrt{(\;125)^2}$

5. Find the value of the following expressions (see 33.7):

a) $\sqrt{2^2 \quad 3^2}$ *b*) $\sqrt{3^4 \quad 5^2}$ *c*) $\sqrt{2^2 \quad 3^2 \quad 3^4}$ *d*) $\sqrt{10^2 \quad 3^6}$
e) $\sqrt{(\;7)^2 \quad 5^4}$ *f*) $\sqrt{(\;5)^4 \quad (\;3)^2}$ *g*) $\sqrt{(\;20)^2 \quad 3^4}$ *h*) $\sqrt{(\;10)^4 \quad (\;5)^6}$

Example:

h) $\sqrt{(\;10)^4 \quad (\;5)^6} \quad \sqrt{10^4 \quad 5^6} \quad 10^{4/2} \quad 5^{6/2} \quad 10^2 \quad 5^3 \quad 100 \quad 125 \quad 12500$

6. Use the prime factorization algorithm to find the following square roots (see 33.7):

a) $\sqrt{1089}$ *b*) $\sqrt{2916}$ *c*) $\sqrt{2500}$ *d*) $\sqrt{15625}$ *e*) $\sqrt{7056}$ *f*) $\sqrt{202500}$

7. Use a calculator to find the following roots. Round the answer to the nearest hundredth (see 33.8):

a) $\sqrt{2}$ *b*) $\sqrt{3}$ *c*) $\sqrt{7}$ *d*) $\sqrt{10}$ *e*) $\sqrt{50}$ *f*) $\sqrt{1234}$ *g*) $\sqrt{99999}$

Example:

f) $\sqrt{99999} \quad 316.23$

8. Use a calculator to find the following roots. Round the answer to the nearest thousandth (see 33.8):

a) $\sqrt{2}$ *b*) $\sqrt{3}$ *c*) $\sqrt{10}$ *d*) $\sqrt{30}$ *e*) $\sqrt{200}$ *f*) $\sqrt{777}$ *g*) $\sqrt{12345}$

34. Cubic Roots

34.1 *(Cubic Roots)* The cubic root operation is the *inverse operation* to raising a number to the power of 3:

$$\sqrt[3]{a} \text{ is that } x \text{ so } x^3 \quad a$$

Example:

$$\sqrt[3]{8} \quad 2; \quad 2^3 \quad 8$$

34.2 *(Useful Cubic Roots)* Here is a list of some basic cubic roots:

$0^3 \quad 0 \quad \sqrt[3]{0} \quad 0; \; 1^3 \quad 1 \quad \sqrt[3]{1} \quad 1$

$2^3 \quad 8 \quad \sqrt[3]{8} \quad 2; \; 3^3 \quad 27 \quad \sqrt[3]{27} \quad 3$

$4^3 \quad 64 \quad \sqrt[3]{64} \quad 4; \; 5^3 \quad 125 \quad \sqrt[3]{125} \quad 5$

$(\;\; 1)^3 \quad 1 \quad \sqrt[3]{\;\;1} \quad 1$

$(\;\; 2)^3 \quad 8 \quad \sqrt[3]{\;\;8} \quad 2$

$(\;\; 3)^3 \quad 27 \quad \sqrt[3]{\;\;27} \quad 3$

34.3 *(The Cubic Root of a Cube)* The cubic root of a number cubed is the original number:

$$\sqrt[3]{a^3} \quad a$$

Examples:

$$\sqrt[3]{3^3} \quad \sqrt[3]{27} \quad 3; \quad \sqrt[3]{(\;\; 2)^3} \quad \sqrt[3]{\;\;8} \quad 2$$

34.4 *(Sign)* The cubic root of a positive number is positive (because a positive number cubed is positive). The cubic root of a negative number is negative (because a negative number cubed is negative).
Examples:

$$\sqrt[3]{125} \quad 5; \quad \sqrt[3]{\;\;125} \quad 5$$

34.5 *(The Algorithm for finding the cubic root)* To find the cubic root of a number:
a) perform the prime factorization
b) the cubic root has the same factors as the original number, but all exponents are *divided by 3*
Example:

$$\sqrt[3]{1728} \quad ?$$

$$1728 \quad 2^6 \quad 3^3$$

$$\sqrt[3]{1728} \quad 2^2 \quad 3^1 \quad 12$$

34.6 *(Real Numbers)* If the radicand of a cubic root is not a perfect cube, the cubic root of that number is a real number. You can use a calculator to find a decimal approximation of that real number. Example:

$$\sqrt[3]{10} \quad 2.154$$

1. Find the cubes of the following numbers:

a) 0 *b)* 1 *c)* 2 *d)* 3 *e)* 4 *f)* 5 *g)* 6

h) 10 *i)* 20 *j)* 50 *k)* 100 *l)* 200 *m)* 11 *n)* 12

2. Find the following cubic roots (use the results you have found at Exercise 1):

a) $\sqrt[3]{0}$ *b)* $\sqrt[3]{1728}$ *c)* $\sqrt[3]{1}$ *d)* $\sqrt[3]{216}$ *e)* $\sqrt[3]{8}$ *f)* $\sqrt[3]{64}$

g) $\sqrt[3]{125}$ *h)* $\sqrt[3]{1000}$ *i)* $\sqrt[3]{27}$ *j)* $\sqrt[3]{8000}$ *k)* $\sqrt[3]{125000}$ *l)* $\sqrt[3]{1331}$

3. Find the cubes of the following numbers:

a) 1 *b)* 2 *c)* 3 *d)* 4 *e)* 5 *f)* 6

g) 10 *h)* 20 *i)* 40 *j)* 100 *k)* 300 *l)* 12

4. Find the following cubic roots (use the results you found in Exercise 3):

a) $\sqrt[3]{\;\;1}$ *b)* $\sqrt[3]{\;\;216}$ *c)* $\sqrt[3]{\;\;8}$ *d)* $\sqrt[3]{216}$ *e)* $\sqrt[3]{\;\;27}$

f) $\sqrt[3]{\;\;1000}$ *g)* $\sqrt[3]{\;\;64}$ *h)* $\sqrt[3]{\;\;8000}$ *i)* $\sqrt[3]{\;\;125}$ *j)* $\sqrt[3]{\;\;1728}$

5. Find the value of the following expressions containing cubic roots of cubes (see 34.3):

a) $\sqrt[3]{1^3}$ *b)* $\sqrt[3]{3^3}$ *c)* $\sqrt[3]{5^3}$ *d)* $\sqrt[3]{10^3}$ *e)* $\sqrt[3]{12^3}$ *f)* $\sqrt[3]{12345^3}$

6. Find the value of the following expressions containing cubic roots of cubes (see 34.3):

a) $\sqrt[3]{(\;\; 1)^3}$ *b)* $\sqrt[3]{(\;\; 2)^3}$ *c)* $\sqrt[3]{(\;\; 4)^3}$ *d)* $\sqrt[3]{(\;\; 10)^3}$ *e)* $\sqrt[3]{(\;\; 123)^3}$

7. Find the value of the following expressions:

a) $\sqrt[3]{2^3 \quad 3^3}$ *b)* $\sqrt[3]{2^3 \quad 5^6}$ *c)* $\sqrt[3]{10^6 \quad 3^3}$ *d)* $\sqrt[3]{2^9 \quad 3^6 \quad 5^3}$

e) $\sqrt[3]{(\;\; 4)^3 \quad 5^6}$ *f)* $\sqrt[3]{(\;\; 5)^6 \quad (\;\; 4)^3}$ *g)* $\sqrt[3]{(\;\; 10)^3 \quad 3^6}$ *h)* $\sqrt[3]{(\;\; 2)^6 \quad (\;\; 10)^3}$

Example:

h) $\sqrt[3]{(\;\; 2)^6 \quad (\;\; 10)^3} \quad (\;\; 2)^{6/3} \quad (\;\; 10)^{3/3} \quad (\;\; 2)^2 \quad (\;\; 10)^1 \quad 4 \quad (\;\; 10) \quad 40$

8. Use the prime factorization algorithm to find the following cubic roots (see 34.5):

a) $\sqrt[3]{5832}$ *b)* $\sqrt[3]{91125}$ *c)* $\sqrt[3]{13824}$ *d)* $\sqrt[3]{32768}$ *e)* $\sqrt[3]{1728000}$

9. Use a calculator to find the following roots. Round the answer to the nearest thousandth (see 34.6):

a) $\sqrt[3]{7}$ *b)* $\sqrt[3]{\;\;20}$ *c)* $\sqrt[3]{80}$ *d)* $\sqrt[3]{100}$ *e)* $\sqrt[3]{\;\;1234}$ *f)* $\sqrt[3]{99999}$

Example:

f) $\sqrt[3]{99999} \quad 46.416$

35. Roots of superior order

35.1 *(n-th Order Roots)* The *n-th* order root operation is the *inverse operation* to raising a number to the power *n*:

$\sqrt[n]{a}$ *is that x so* $x^n \quad a$

Example:

$\sqrt[4]{16} \quad 2; \quad 2^4 \quad 16$

$\sqrt[5]{\quad 32} \quad 2; \quad (\quad 2)^5 \quad 32$

35.2 *(Radicand)* Because any real number (including an integer) raised to an *even power* is positive or 0 the radicand must be positive or 0:

$\sqrt[n]{a}; \quad a \quad 0$ *if n is even*

Example:

$\sqrt[4]{\quad 81}$ *is not real because* $81 \quad 0$

35.3 *(The Principal Root)* If *n is even*, more numbers raised to the power of *n* are equal to the radicand. By convention the *n-th* root is represented by the positive number called the *principal root*:

$\sqrt[n]{a} \quad 0$ *if n is even*

Example:

$3^4 \quad 81; (\quad 3)^4 \quad 81; \quad \sqrt[4]{81} \quad 3$

35.4 *(Basic n-th order roots)* Here is a list of some basic *n-th* order roots:

$0^4 \quad 0 \quad \sqrt[4]{0} \quad 0; \quad 1^5 \quad 1 \quad \sqrt[5]{1} \quad 1$

$2^4 \quad 16 \quad \sqrt[4]{16} \quad 2; \quad 3^4 \quad 81 \quad \sqrt[4]{81} \quad 3$

$(\quad 2)^7 \quad 18 \quad \sqrt[7]{\quad 128} \quad 2$

35.5 *(The n-th order root of a positive number raised to power n)* The *n-th* order root of a positive number raised to power *n* is the original number:

$\sqrt[n]{a^n} \quad a; \quad a \quad 0$

Examples:

$\sqrt[5]{3^5} \quad 5; \quad \sqrt[6]{10^6} \quad 10$

35.6 *(The n-th order root of a negative number raised to power n)* The *n-th* order root of a negative number raised to power *n* is the original number if *n* is odd and the opposite of the original number if *n* is even.

Examples:

$\sqrt[5]{(\quad 2)^5} \quad 2; \quad \sqrt[8]{(\quad 2)^8} \quad \sqrt[8]{256} \quad \sqrt[8]{2^8} \quad 2$

35.7 *(The Algorithm for finding the n-th root)* To find the *n-th* root of a number:

a) perform the prime factorization

b) the *n-th* root has the same factors as the original number, but all exponents are divided by *n*

Example:

$\sqrt[4]{20736} \quad \sqrt[4]{2^8 \quad 3^4} \quad 2^2 \quad 3^1 \quad 12$

1. Find the values of the following powers:

a) 0^5 *b)* 2^4 *c)* 3^4 *d)* 4^5 *e)* 5^4 *f)* 10^6 *g)* 100^4

h) $(\quad 1)^5$ *i)* $(\quad 1)^6$ *j)* $(\quad 2)^4$ *k)* $(\quad 2)^5$ *l)* $(\quad 3)^4$ *m)* $(\quad 10)^4$ *n)* $(\quad 10)^5$

2. Find the value of the following roots (use the results you found in Exercise 1):

a) $\sqrt[5]{0}$ *b)* $\sqrt[5]{\quad 1}$ *c)* $\sqrt[4]{16}$ *d)* $\sqrt[4]{625}$ *e)* $\sqrt[6]{1000000}$ *f)* $\sqrt[4]{81}$

g) $\sqrt[7]{\quad 1}$ *h)* $\sqrt[5]{\quad 32}$ *i)* $\sqrt[5]{\quad 100000}$ *j)* $\sqrt[4]{\quad 1}$ *k)* $\sqrt[5]{\quad 243}$ *l)* $\sqrt[9]{\quad 512}$

Example:

i) $\sqrt[5]{\quad 100000} \quad 10 \quad (\quad 10)^5 \quad 100000$

3. Find the value of the following expressions containing roots of powers (see 35.5):

a) $\sqrt[5]{1^5}$ *b)* $\sqrt[4]{4^4}$ *c)* $\sqrt[7]{2^7}$ *d)* $\sqrt[9]{10^9}$ *e)* $\sqrt[7]{12^7}$ *f)* $\sqrt[11]{54321^{11}}$

Example:

f) $\sqrt[11]{54321^{11}} \quad 54321$

4. Find the value of the following expressions containing roots of powers (see 35.6):

a) $\sqrt[6]{(\quad 1)^6}$ *b)* $\sqrt[5]{(\quad 1)^5}$ *c)* $\sqrt[7]{(\quad 4)^7}$ *d)* $\sqrt[5]{(\quad 10)^5}$ *e)* $\sqrt[10]{(\quad 123)^{10}}$

Example:

e) $\sqrt[10]{(\quad 123)^{10}} \quad \sqrt[10]{123^{10}} \quad 123$

5. Find the value of the following expressions (see 35.7):

a) $\sqrt[5]{2^5 \quad 3^{10}}$ *b)* $\sqrt[4]{2^4 \quad 5^{12}}$ *c)* $\sqrt[4]{10^8 \quad 3^4}$ *d)* $\sqrt[5]{2^{10} \quad 3^{10} \quad 3^5}$

e) $\sqrt[4]{(\quad 4)^8 \quad 5^4}$ *f)* $\sqrt[5]{(\quad 5)^5 \quad (\quad 4)^{10}}$ *g)* $\sqrt[7]{(\quad 10)^7 \quad 3^{14}}$ *h)* $\sqrt[5]{(\quad 2)^{15} \quad (\quad 10)^{10}}$

Example:

h) $\sqrt[5]{(\quad 2)^{15} \quad (\quad 10)^{10}} \quad (\quad 2)^{15/5} \quad (\quad 10)^{10/5} \quad (\quad 2)^3 (\quad 10)^2 \quad 8 \quad 100 \quad 800$

6. Use the prime factorization algorithm to find the following roots (see 35.7):

a) $\sqrt[4]{20736}$ *b)* $\sqrt[5]{59049}$ *c)* $\sqrt[6]{46656}$ *d)* $\sqrt[6]{4096}$ *e)* $\sqrt[7]{78125}$

7. Use a calculator to find the following roots. Round the answer to the nearest hundredth:

a) $\sqrt[4]{10}$ *b)* $\sqrt[5]{100}$ *c)* $\sqrt[6]{125}$ *d)* $\sqrt[5]{\quad 100}$ *e)* $\sqrt[4]{100}$ *f)* $\sqrt[7]{\quad 12345}$

Example:

f) $\sqrt[7]{\quad 12345} \quad 3.84$

36. Roots Rules

36.1 *(Product Rule)* The product of two or more like roots is equal to the root of the product of the radicands:

$$\sqrt[n]{a}\ \ \sqrt[n]{b}\ \ \ldots\ \ \sqrt[n]{a\ \ b\ \ \ldots}$$

Examples:

$$\sqrt[3]{2}\ \ \sqrt[3]{4}\ \ \sqrt[3]{2\ \ 4}\ \ \sqrt[3]{8}\ \ 2$$

$$\sqrt{10}\ \ \sqrt{10}\ \ \sqrt{10\ \ 10}\ \ \sqrt{10^2}\ \ 10$$

36.2 *(Simplifying Radicals)* The product rule allows you to simplify radicals. Example:

$$\sqrt{50}\ \ \sqrt{25\ \ 2}\ \ \sqrt{25}\ \ \sqrt{2}\ \ 5\ \ \sqrt{2}\ \ 5\sqrt{2}$$

36.3 *(Quotient Rule)* The ratio of two like roots is equal to the root of the ratio of the radicands:

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}\ \ \sqrt[n]{\frac{a}{b}}$$

Example:

$$\frac{\sqrt[3]{128}}{\sqrt[3]{2}}\ \ \sqrt[3]{\frac{128}{2}}\ \ \sqrt[3]{64}\ \ 4$$

36.4 *(Simplifying Radicals)* The quotient rule allows you to simplify radicals. Example:

$$\sqrt{\frac{25}{4}}\ \ \frac{\sqrt{25}}{\sqrt{4}}\ \ \frac{5}{2}$$

36.5 *(Power Rule)* A root raised to a power is equal to the root of the radicand raised to the same power:

$$(\sqrt[n]{a})^m\ \ \sqrt[n]{a^m}$$

Example:

$$(\sqrt[3]{5})^2\ \ \sqrt[3]{5^2}\ \ \sqrt[3]{25}$$

36.6 *(Canceling Out)* The *n-th* order root raised to the power of *n* is equal to the radicand:

$$(\sqrt[n]{a})^n\ \ a$$

Examples:

$$(\sqrt{3})^2\ \ 3;\ \ (\sqrt[5]{\ \ 32})^5\ \ 32$$

36.7 *(Simplifying Expressions)* You may use all the above rules to simplify expressions containing radicals. Example:

$${\frac{\sqrt{2}\ \ \sqrt{15}}{\sqrt{5}}}^{2}\ \ {\frac{\sqrt{2\ \ 15}}{\sqrt{5}}}^{2}\ \ {\frac{\sqrt{30}}{\sqrt{5}}}^{2}$$

$${\sqrt{\frac{30}{5}}}^{2}\ \ (\sqrt{6})^2\ \ 6;\ \ or:$$

$${\frac{\sqrt{2}\ \ \sqrt{15}}{\sqrt{5}}}^{2}\ \ \frac{(\sqrt{2})^2\ \ (\sqrt{15})^2}{(\sqrt{5})^2}\ \ \frac{2\ \ 15}{5}\ \ 6$$

1. Use the product rule (see 36.1) to simplify the following expressions:

a) $\sqrt{2}\ \ \sqrt{2}$ *b)* $\sqrt[3]{4}\ \ \sqrt[3]{4}\ \ \sqrt[3]{4}$ *c)* $\sqrt[3]{2}\ \ \sqrt[3]{4}$ *d)* $\sqrt[5]{\ \ 2}\ \ \sqrt[5]{16}$

e) $\sqrt[4]{4}\ \ \sqrt[4]{4}$ *f)* $\sqrt[3]{\ \ 5}\ \ \sqrt[3]{\ \ 5}\ \ \sqrt[3]{\ \ 5}$ *g)* $\sqrt[7]{\ \ 10^3}\ \ \sqrt[7]{10^4}$ *h)* $\sqrt{2^{\ \ 1}}\ \ \sqrt{8}$

2. Use the product rule to simplify the following radicals (see 36.2):

a) $\sqrt{8}$ *b)* $\sqrt{18}$ *c)* $\sqrt[5]{128}$ *d)* $\sqrt[3]{\ \ 54}$ *e)* $\sqrt[3]{32}$ *f)* $\sqrt[4]{10^6}$

g) $\sqrt{200}$ *h)* $\sqrt[3]{\ \ 10^4}$ *i)* $\sqrt[5]{10^6}$ *j)* $\sqrt{128}$ *k)* $\sqrt[3]{\ \ 32}$ *l)* $\sqrt[3]{500}$

3. Use the quotient rule (see 36.3) to simplify the following expressions:

a) $\frac{\sqrt{50}}{\sqrt{2}}$ *b)* $\frac{\sqrt[3]{16}}{\sqrt[3]{2}}$ *c)* $\frac{\sqrt{250}}{\sqrt{2}}$ *d)* $\frac{\sqrt[3]{\ \ 500}}{\sqrt[3]{4}}$ *e)* $\frac{\sqrt[3]{\ \ 10^4}}{\sqrt[3]{\ \ 10}}$ *f)* $\frac{\sqrt{3}}{\sqrt{75}}$

g) $\frac{\sqrt[3]{3}}{\sqrt[3]{24}}$ *h)* $\frac{\sqrt[5]{128}}{\sqrt[5]{\ \ 4}}$ *i)* $\frac{\sqrt[5]{96}}{\sqrt[5]{3}}$ *j)* $\frac{\sqrt[4]{4}}{\sqrt[4]{64}}$ *k)* $\frac{\sqrt{15}}{\sqrt{120}}$ *l)* $\frac{\sqrt[3]{\ \ 4}}{\sqrt[3]{500}}$

4. Use the quotient rule to simplify the following expressions (see 36.4):

a) $\sqrt{\frac{9}{25}}$ *b)* $\sqrt[3]{\frac{8}{27}}$ *c)* $\sqrt[5]{\frac{32}{243}}$ *d)* $\sqrt[3]{\frac{125}{64}}$ *e)* $\sqrt[3]{\frac{1000}{27}}$ *f)* $\sqrt[4]{\frac{625}{81}}$

5. Use the power rule (see 36.5) to rewrite (simplify) the following expressions:

a) $(\sqrt{2})^3$ *b)* $(\sqrt[3]{5})^2$ *c)* $(\sqrt{3})^5$ *d)* $(\sqrt[3]{5})^4$ *e)* $(\sqrt[4]{10})^2$ *f)* $(\sqrt[3]{\ \ 5})^2$

Example:

d) $(\sqrt[3]{5})^4\ \ \sqrt[3]{5^4}\ \ \sqrt[3]{5^3\ \ 5}\ \ \sqrt[3]{5^3}\ \ \sqrt[3]{5}\ \ 5\ \ \sqrt[3]{5}$

6. Use the canceling out rule (see 36.6) to simplify the following expressions:

a) $(\sqrt{2})^2$ *b)* $(\sqrt[3]{5})^3$ *c)* $(\sqrt{20})^2$ *d)* $(\sqrt[3]{\ \ 10})^3$ *e)* $(\sqrt[4]{10})^4$ *f)* $(\sqrt[5]{\ \ 5})^5$

7. Simplify the following expressions with radicals:

a) $\frac{\sqrt{3}\ \ \sqrt{6}}{\sqrt{2}}$ *b)* $\frac{\sqrt{3}}{\sqrt{30}\ \ \sqrt{10}}$ *c)* $\frac{\sqrt[3]{\ \ 2}\ \ \sqrt[3]{\ \ 20}}{\sqrt[3]{\ \ 5}}$ *d)* $\frac{\sqrt[3]{8}\ \ \sqrt[3]{10}}{\sqrt[3]{2}\ \ \sqrt[3]{5}}$ *e)* $\frac{\sqrt[5]{2}\ \ \sqrt[5]{16}}{\sqrt[5]{32}\ \ \sqrt[5]{32}}$

f) ${\frac{\sqrt{3}}{\sqrt{2}}}^{2}$ *g)* ${\frac{\sqrt[3]{\ \ 2}}{\sqrt[3]{\ \ 5}}}^{3}$ *h)* ${\frac{\sqrt{2}\ \ \sqrt{3}}{\sqrt{5}}}^{2}$ *i)* ${\frac{\sqrt[3]{2}\ \ \sqrt[3]{3}}{\sqrt[3]{4}\ \ \sqrt[3]{5}}}^{3}$ *j)* $\frac{\sqrt{3}}{\sqrt{2}}\ \ \frac{\sqrt{3}}{\sqrt{2}}$

k) $\frac{\sqrt{2}\ \ \sqrt{2}\ \ \sqrt{3}}{\sqrt{6}\ \ \sqrt{8}}$ *l)* $\frac{\sqrt{5}\ \ \sqrt{10}}{\sqrt{150}\ \ \sqrt{2}\ \ \sqrt{6}}$ *m)* $\frac{\sqrt[3]{2}\ \ \sqrt[3]{\ \ 3}}{\sqrt[3]{40}\ \ \sqrt[3]{25}\ \ \sqrt[3]{6}}$

n) $\frac{\sqrt{2}}{\sqrt{60}}\ \ \frac{\sqrt{3}}{\sqrt{10}}\ \ \frac{\sqrt{5}}{\sqrt{20}}$ *o)* ${\frac{\sqrt{1}\ \ \sqrt{2}\ \ \sqrt{3}}{\sqrt{4}\ \ \sqrt{5}\ \ \sqrt{6}}}^{2}$ *p)* ${\frac{\sqrt{3}\ \ \sqrt[3]{2}}{\sqrt[3]{16}\ \ \sqrt{5}}}^{2}$

Example:

k) $\frac{\sqrt{2}\ \ \sqrt{2}\ \ \sqrt{3}}{\sqrt{6}\ \ \sqrt{8}}\ \ \frac{\sqrt{2\ \ 2\ \ 3}}{\sqrt{6\ \ 8}}\ \ \frac{\sqrt{12}}{\sqrt{48}}\ \ \sqrt{\frac{12}{48}}\ \ \sqrt{\frac{1}{4}}\ \ \frac{\sqrt{1}}{\sqrt{4}}\ \ \frac{1}{2}$

37. Equations with powers and radicals

37.1 *(Roots)* Consider the equation:

$$x^n \quad a$$

Any number x satisfying the equation is called a *root* of the equation. Example:

$$x^2 \quad 4$$

The two roots of this equations are:

$$x \quad 2 \text{ because } (\ 2)^2 \quad 4$$

37.2 *(n is even)* In the case that *n is even* (e.g. 2, 4, 6, ...) the equation:

$$x^n \quad a;\ a \quad 0$$

has two real roots given by:

$$x \quad \sqrt[n]{a}$$

There are *no real roots* if *a<0*. Examples:

$$x^4 \quad 81 \quad x \quad \sqrt[4]{81} \quad 3$$

$$x^2 \quad 1 \quad \text{no real roots}$$

37.3 *(n is odd)* In the case that *n is odd* (e.g. 3, 5, 7, ...) the equation:

$$x^n \quad a$$

has always a real solution given by:

$$x \quad \sqrt[n]{a}$$

Examples:

$$x^3 \quad 27 \quad x \quad \sqrt[3]{\ 27} \quad 3$$

$$x^5 \quad 32 \quad x \quad \sqrt[5]{32} \quad 2$$

37.4 *(Unknown Radicand)* To find the solution of the equation:

$$\sqrt[n]{x} \quad a$$

raise left and right sides to power n:

$$(\sqrt[n]{x})^n \quad a^n \quad x \quad a^n$$

Examples:

$$\sqrt{x} \quad 5 \quad x \quad 5^2 \quad 25$$

$$\sqrt[3]{x} \quad 2 \quad x \quad (\ 2)^3 \quad 8$$

37.5 *(More Steps Solution)* You may need to do more steps in order to solve an equation with powers and radicals. Example:

$$(\sqrt{x^3 \quad 8} \quad 1)^3 \quad 27;$$

$$\sqrt{x^3 \quad 8} \quad 1 \quad \sqrt[3]{27} \quad 3$$

$$\sqrt{x^3 \quad 8} \quad 3 \quad 1 \quad 4$$

$$x^3 \quad 8 \quad 4^2 \quad 16$$

$$x^3 \quad 16 \quad 8 \quad 8$$

$$x \quad \sqrt[3]{8} \quad 2$$

1. Check by substitution if the given x is a solution of the given equation:

$a)\ x \quad 3; x^2 \quad 9$ $b)\ x \quad 4; x^2 \quad 16$ $c)\ x \quad 4; x^3 \quad 64$

$d)\ x \quad 1; x^7 \quad 1$ $e)\ x \quad 5; x^3 \quad 125$ $f)\ x \quad 6; x^3 \quad 216$

$g)\ x \quad 5; x^5 \quad 125$ $h)\ x \quad 10; x^4 \quad 10000$ $i)\ x \quad 2; x^{10} \quad 1024$

2. Solve the following equations (see 37.2):

$a)\ x^2 \quad 25$ $b)\ x^2 \quad 4$ $c)\ x^4 \quad 625$ $d)\ x^2 \quad 64$ $e)\ x^4 \quad 10000$

$f)\ x^8 \quad 256$ $g)\ x^4 \quad 81$ $h)\ x^6 \quad 64$ $i)\ x^4 \quad 625$ $j)\ x^{10} \quad 1024$

3. Solve the following equations (see 37.3):

$a)\ x^3 \quad 64$ $b)\ x^5 \quad 32$ $c)\ x^3 \quad 125$ $d)\ x^7 \quad 128$ $e)\ x^3 \quad 1000$

$f)\ x^5 \quad 32$ $g)\ x^3 \quad 64$ $h)\ x^7 \quad 128$ $i)\ x^3 \quad 216$ $j)\ x^5 \quad 100000$

4. Solve for x (see 37.4):

$a)\ \sqrt{x} \quad 2$ $b)\ \sqrt[3]{x} \quad 2$ $c)\ \sqrt{x} \quad 3$ $d)\ \sqrt[4]{x} \quad 2$ $e)\ \sqrt{x} \quad 1$

$f)\ \sqrt[3]{x} \quad 3$ $g)\ \sqrt[3]{x} \quad 2$ $h)\ \sqrt[3]{x} \quad 1$ $i)\ \sqrt[3]{x} \quad 5$ $j)\ \sqrt[4]{x} \quad 10$

5. Solve for x (see 37.4):

$a)\ \sqrt[x]{25} \quad 5$ $b)\ \sqrt[x]{8} \quad 2$ $c)\ \sqrt[x]{\ 1} \quad 1$ $d)\ \sqrt[x]{64} \quad 2$

$e)\ \sqrt[x]{\ 125} \quad 5$ $f)\ \sqrt[x]{1024} \quad 2$ $g)\ \sqrt[x]{\ 64} \quad 4$ $h)\ \sqrt[x]{27} \quad 3$

Example:

$$g)\ \sqrt[x]{\ 64} \quad 4 \quad (\ 4)^x \quad 64 \quad (\ 4)^x \quad (\ 4)^3 \quad x \quad 3$$

6. Find x that satisfies the following equations:

$a)\ \sqrt[x]{4} \quad x$ $b)\ \sqrt[x]{27} \quad x$ $c)\ \sqrt[x]{256} \quad x$ $d)\ \sqrt[x]{3125} \quad x$ $e)\ \sqrt[x]{46656} \quad x$

Example:

$$e)\ \sqrt[x]{46656} \quad x \quad x^x \quad 46656 \quad 6^6 \quad 46656 \quad x \quad 6$$

7. Solve for x (see 37.5):

$a)\ \sqrt{x \quad 1} \quad 2$ $b)\ \sqrt[3]{2 \quad x \quad 4} \quad 2$ $c)\ \sqrt{3 \quad (x \quad 2)} \quad 3$

$d)\ \sqrt[3]{2 \quad (x^3 \quad 5)} \quad \sqrt{16}$ $e)\ \sqrt[3]{\dfrac{x^3}{3} \quad 1} \quad \sqrt[3]{\ 8}$ $f)\ (\sqrt[3]{x} \quad 3)^2 \quad (\ 5)^2$

$g)\ \sqrt[4]{2 \quad x^2 \quad 2} \quad 2$ $h)\ \sqrt[5]{5 \quad (x^3 \quad 1) \quad 3} \quad \sqrt[3]{\ 8}$ $i)\ \dfrac{\sqrt{x} \quad \sqrt{4}}{\sqrt{25} \quad \sqrt{4}} \quad \dfrac{\sqrt[3]{64}}{\sqrt[3]{8}}$

$j)\ \dfrac{\sqrt{5}}{\sqrt[3]{x} \quad \sqrt[3]{\ 8}} \quad \dfrac{1}{\sqrt{5}}$ $k)\ (\sqrt{x} \quad 1) \quad (\sqrt{4} \quad 1) \quad \sqrt[3]{\ 27} \quad 0$ $l)\ \dfrac{\sqrt[4]{3} \quad \sqrt[3]{10}}{\sqrt[3]{x} \quad \sqrt{3}} \quad \dfrac{\sqrt{3}}{\sqrt[4]{27}}$

38. Link between Radicals and Powers

38.1 *(The Link between Radicals and Powers)* If the *radicand is positive*, a radical expression can be converted to a power according to the following formula:

$$\sqrt[n]{a} \quad a^{\frac{1}{n}}; \quad a \quad 0$$

Example:

$$\sqrt{3} \quad 3^{1/2}$$

38.2 *(Generalization)* The precedent relation can be generalized to:

$$\sqrt[n]{a^m} \quad a^{\frac{m}{n}}; \quad a \quad 0$$

Example:

$$\sqrt[3]{5^2} \quad 5^{2/3}$$

38.3 *(Simplifying Radicals)* The following formula allows you to simplify radicals:

$$\sqrt[m\ n]{a^{m\ k}} \quad a^{\frac{m\ k}{m\ n}} \quad a^{\frac{k}{n}} \quad \sqrt[n]{a^k}$$

Example:

$$\sqrt[6]{5^3} \quad \sqrt[2]{5} \quad \sqrt{5}$$

38.4 *(Multiplying Unlike Radicals)* Using the link between radicals and powers, we can now multiply unlike radicals:

$$\sqrt[m]{a} \quad \sqrt[n]{a} \quad a^{\frac{1}{m}} \quad a^{\frac{1}{n}} \quad a^{\frac{1}{m}\ \frac{1}{n}} \quad a^{\frac{m\ n}{m\ n}}$$

$$\sqrt[m]{a} \quad \sqrt[n]{a} \quad \sqrt[m\ n]{a^{m\ n}}$$

Example:

$$\sqrt{2} \quad \sqrt[3]{2} \quad \sqrt[2\ 3]{2^{2\ 3}} \quad \sqrt[6]{2^5} \quad \sqrt[6]{32}$$

38.5 *(Dividing Unlike Radicals)* Using the link between radicals and powers, we can now divide unlike radicals:

$$\frac{\sqrt[m]{a}}{\sqrt[n]{a}} \quad \frac{a^{\frac{1}{m}}}{a^{\frac{1}{n}}} \quad a^{\frac{1}{m}\ \frac{1}{n}} \quad a^{\frac{n\ m}{n\ m}} \quad \sqrt[n\ m]{a^{n\ m}}$$

Example:

$$\frac{\sqrt[3]{\ 8}}{\sqrt[5]{\ 8}} \quad \frac{\sqrt[3]{8}}{\sqrt[5]{8}} \quad \frac{\sqrt[3]{8}}{\sqrt[5]{8}} \quad \sqrt[3\ 5]{8^{5\ 3}} \quad \sqrt[15]{8^2}$$

$$\sqrt[15]{2^6} \quad \sqrt[5]{2^2} \quad \sqrt[5]{4}$$

38.6 *(Radical of Radicals)* If the radicand of a radical is another radical, you can use the following formula to simplify the expression:

$$\sqrt[m]{\sqrt[n]{a}} \quad \sqrt[m\ n]{a}$$

Indeed:

$$\sqrt[m]{\sqrt[n]{a}} \quad (\sqrt[n]{a})^{\frac{1}{m}} \quad (a^{\frac{1}{n}})^{\frac{1}{m}} \quad a^{\frac{1}{m\ n}} \quad \sqrt[m\ n]{a}$$

Example:

$$\sqrt[3]{\sqrt{64}} \quad \sqrt[3\ 2]{64} \quad \sqrt[6]{64} \quad 2$$

1. Convert these radical notations into exponential notations:

a) $\sqrt{2}$ *b)* $\sqrt[3]{27}$ *c)* $\sqrt[4]{5}$ *d)* $\sqrt[5]{2}$ *e)* $\sqrt[4]{64}$ *f)* $\sqrt[10]{1024}$ *g)* $\sqrt[5]{\ 2}$

2. Convert these exponential notations into radical notations:

a) $2^{1/2}$ *b)* $3^{1/2}$ *c)* $8^{1/3}$ *d)* $10^{1/2}$ *e)* $27^{1/3}$ *f)* $100^{1/2}$ *g)* $625^{1/4}$

3. Convert these radical notations into exponential notations:

a) $\sqrt{2^3}$ *b)* $\sqrt[3]{7^2}$ *c)* $\sqrt[5]{5^2}$ *d)* $\sqrt[6]{10^5}$ *e)* $\sqrt[7]{7^3}$ *f)* $\sqrt[3]{3^6}$ *g)* $\sqrt[3]{2^2}$

4. Convert these exponential notations into radical notations:

a) $3^{3/5}$ *b)* $5^{3/2}$ *c)* $10^{5/4}$ *d)* $9^{2/3}$ *e)* $7^{4/3}$ *f)* $20^{4/5}$ *g)* $5^{5/3}$

5. Simplify the following radicals (see 38.3):

a) $\sqrt{3^4}$ *b)* $\sqrt[3]{2^9}$ *c)* $\sqrt[4]{25^2}$ *d)* $\sqrt[3]{2^9}$ *e)* $\sqrt[5]{2^{10}}$ *f)* $\sqrt[10]{2^5}$ *g)* $\sqrt[6]{27^3}$

6. Write the following products of radicals as unique radicals (see 38.4):

a) $\sqrt{3} \quad \sqrt[3]{3}$ *b)* $\sqrt[3]{10} \quad \sqrt[4]{10}$ *c)* $\sqrt[3]{5} \quad \sqrt[5]{5}$ *d)* $\sqrt{2} \quad \sqrt[4]{2}$

e) $\sqrt[4]{2} \quad \sqrt[5]{2}$ *f)* $\sqrt{10} \quad \sqrt[3]{10} \quad \sqrt[4]{10}$ *g)* $\sqrt{2} \quad \sqrt[4]{2} \quad \sqrt[6]{2}$ *h)* $\sqrt{3} \quad \sqrt[3]{3} \quad \sqrt[4]{3} \quad \sqrt[5]{3}$

Example:

f) $\sqrt{10} \quad \sqrt[3]{10} \quad \sqrt[4]{10} \quad 10^{\frac{1}{2}} \quad 10^{\frac{1}{3}} \quad 10^{\frac{1}{4}} \quad 10^{\frac{1}{2}\ \frac{1}{3}\ \frac{1}{4}} \quad 10^{\frac{6}{12}\ \frac{4}{12}\ \frac{3}{12}} \quad 10^{\frac{13}{12}}$

$10^{1\ \frac{1}{12}} \quad 10 \quad \sqrt[12]{10}$

7. Write the following ratios of radicals as unique radicals (see 38.5):

a) $\frac{\sqrt{2}}{\sqrt[3]{2}}$ *b)* $\frac{\sqrt[3]{10}}{\sqrt[4]{10}}$ *c)* $\frac{\sqrt[3]{5}}{\sqrt{5}}$ *d)* $\frac{\sqrt[5]{2}}{\sqrt[4]{2}}$ *e)* $\frac{\sqrt[4]{5}}{\sqrt[6]{5}}$ *f)* $\frac{\sqrt[6]{10}}{\sqrt[3]{10}}$ *g)* $\frac{\sqrt{3}}{\sqrt[4]{3}}$

8. Write the following expressions with radicals as unique radicals (see 38.6):

a) $\sqrt{\sqrt[3]{2}}$ *b)* $\sqrt[3]{\sqrt{2}}$ *c)* $\sqrt[4]{\sqrt[3]{10}}$ *d)* $\sqrt{\sqrt{5}}$ *e)* $\sqrt[3]{\sqrt[3]{10}}$ *f)* $\sqrt[5]{\sqrt[3]{4}}$ *g)* $\sqrt[6]{\sqrt{64}}$

9. Write the following expressions with radicals as unique radicals (one case is solved for you as an example):

a) $\frac{\sqrt{2}}{\sqrt[3]{2}} \quad \sqrt[4]{2}$ *b)* $\frac{\sqrt{10} \quad \sqrt[3]{10}}{\sqrt[4]{10}}$ *c)* $\frac{\sqrt{3} \quad \sqrt[5]{3}}{\sqrt[4]{3} \quad \sqrt[3]{3}}$ *d)* $\frac{\sqrt[3]{2}}{\sqrt{2}} \quad \frac{\sqrt[5]{2}}{\sqrt[4]{2}}$ *e)* $\frac{\sqrt{3}}{\sqrt[3]{3^2}}$

f) $\sqrt{\sqrt{\sqrt{2}}}$ *g)* $\sqrt{\sqrt[3]{\sqrt{10}}}$ *h)* $\frac{\sqrt[3]{\sqrt{5}}}{\sqrt{5}}$ *i)* $\sqrt[3]{\sqrt{5}} \quad \sqrt{5}$ *j)* $\frac{\sqrt[3]{\sqrt{5}}}{\sqrt[4]{\sqrt{5}}}$

Example:

a) $\frac{\sqrt{2}}{\sqrt[3]{2}} \quad \sqrt[4]{2} \quad \frac{2^{\frac{1}{2}}}{2^{\frac{1}{3}}} \quad 2^{\frac{1}{4}} \quad 2^{\frac{1}{2}\ \frac{1}{3}} \quad 2^{\frac{1}{4}} \quad 2^{\frac{1}{6}} \quad 2^{\frac{1}{4}} \quad 2^{\frac{1}{6}\ \frac{1}{4}} \quad 2^{\frac{2}{12}\ \frac{3}{12}} \quad 2^{\frac{5}{12}} \quad \sqrt[12]{2^5}$

39. Order of Operations (V)

39.1 *(Radical Symbol)* The radical symbol is also considered a grouping symbol:

$$\sqrt[n]{a} \qquad \sqrt[(n)]{(a)}$$

So, to evaluate an expression containing a radical symbol, follow these steps:
a) evaluate the order of the radical (n). This must be a positive integer.
b) evaluate the radicand (a).
c) evaluate the value of the radical.
Example:

$$\sqrt[2\ 3\ 4]{2\ \ 5\ \ 6}\ \ \sqrt[2]{2\ \ 5\ \ 6}\ \ \sqrt{16}\ \ 4$$

39.2 *(Nested Radicals)* If the radicand of a radical contains other radical symbols, start the evaluation of the expression with the innermost radical. Example:

$$\sqrt{5\ \ \sqrt[3]{\ \ 64}}\ \ \sqrt{5\ \ (\ \ 4)}\ \ \sqrt{9}\ \ 3$$

39.3 *(Order of Operations)* If an expression contains all kind of operations including radicals, then consider radicals as independent units. Evaluate the radicals and substitute their values in the original expression. All previously explained rules for the order of operations still apply.
Examples:

1. $\sqrt{4}\ \ \sqrt[3]{\ \ 27}\ \ 2\ \ (\ \ 3)\ \ 5$

2. $\dfrac{\sqrt[3]{\ \ 8}}{\sqrt{4}}\ \ (\sqrt{25}\ \ \sqrt[5]{\ \ 32})\ \ \sqrt{(\ \ 2)\ \ (\ \ 2)}$
$\dfrac{\ \ 2}{2}\ \ (5\ \ (\ \ 2))\ \ \sqrt{4}\ \ \ 1\ \ 3\ \ 2\ \ \ 7$

3. $\sqrt{(\sqrt{16}\ \ 2)^2\ \ (1\ \ \sqrt[3]{\ \ 8})^2}$
$\sqrt{(4\ \ 2)^2\ \ (1\ \ (\ \ 2))^2}\ \ \sqrt{4\ \ 9}\ \ \sqrt{13}$

4. $(\sqrt{2}\ \ \sqrt{3})\ \ (\sqrt{2}\ \ \sqrt{3})\ \ \sqrt{2}\ \ \sqrt{2}$
$\sqrt{2}\ \ \sqrt{2}\ \ \sqrt{3}\ \ \sqrt{2}\ \ \sqrt{3}\ \ \sqrt{3}$
$2\ \ \sqrt{6}\ \ \sqrt{6}\ \ 3\ \ 2\ \ 3\ \ \ 1$

5. $\dfrac{\sqrt{25}\ \ \sqrt{16}}{\sqrt[3]{27}\ \ \sqrt[3]{\ \ 64}}\ \ \sqrt[3]{(\sqrt{(\ \ 3)^4}\ \ \sqrt[3]{\ \ 125})^6}$
$\dfrac{5\ \ 4}{3\ \ 4}\ \ (9\ \ 5)^2\ \ (\ \ 1)\ \ 16\ \ \ 16$

6. $(\sqrt{4})^{\sqrt[3]{\ \ 8}}\ \ (\sqrt[3]{\ \ 27})^{\sqrt{9}}\ \ 2^{\ \ 3}\ \ (\ \ 3)^3$
$\dfrac{1}{2^3}\ \ (\ \ 27)\ \ \dfrac{1}{8}\ \ (\ \ 27)\ \ \ \dfrac{27}{8}$

1. Find the value of each expression:

$a)\ \ \sqrt[(\ \ 1)\ \ (\ \ 3)]{(\ \ 2)^3\ \ 4\ \ 5}$ $\quad b)\ \ \sqrt[2^2]{(\ \ 3)^2\ \ 7}$ $\quad c)\ \ \sqrt[\frac{5}{2}\ \ (\ \ 2)^2]{(2^5)^4}$

$d)\ \ \sqrt[(\ \ 8)\ \ (\ \ 4)]{(\ \ 8)\ \ (\ \ 2)}$ $\quad e)\ \ \sqrt[(\sqrt[5]{32})^2\ \ 1]{\ \ 2\ \ [(\ \ 3)^3\ \ (\ \ 2)^3]\ \ 6}$ $\quad f)\ \ \sqrt[\sqrt{4}]{1\ \ \dfrac{(\ \ 2)^3}{(\ \ 1)^2}}$

Example:

$f)\ \ \sqrt[\sqrt{4}]{1\ \ \dfrac{(\ \ 2)^3}{(\ \ 1)^2}}\ \ \sqrt[2]{1\ \ \dfrac{\ \ 8}{1}}\ \ \sqrt{1\ \ 8}\ \ \sqrt{9}\ \ 3$

2. Find the value of each expression (see 39.2):

$a)\ \ 1\ \ \sqrt[3]{\sqrt{3^4\ \ 2^4\ \ 1}}$ $\quad b)\ \ \sqrt[4]{(1\ \ \sqrt{4})^2\ \ (1\ \ \sqrt{4})^2\ \ 7}$ $\quad c)\ \ \sqrt[3]{2\ \ (\sqrt{4}\ \ \sqrt[3]{\ \ 8})}$

$d)\ \ \sqrt{(\ \ 2)\ \ \sqrt[3]{\ \ \sqrt{64}}}$ $\quad e)\ \ (\ \ \sqrt[3]{\ \ 27}\ \ \sqrt{25})^{\sqrt{9}}$ $\quad f)\ \ \sqrt[\sqrt[3]{8}]{\sqrt[3]{64}}\ \ \sqrt[\sqrt[3]{27}]{\sqrt{64}}$

Example:

$f)\ \ \sqrt[\sqrt[3]{8}]{\sqrt[3]{64}}\ \ \sqrt[\sqrt[3]{27}]{\sqrt{64}}\ \ \sqrt[2]{4}\ \ \sqrt[3]{\sqrt{64}}\ \ 2\ \ \sqrt{4}\ \ 2\ \ 2\ \ 0$

3. Use the order of operations to find the value of each expression:

$a)\ \ \sqrt{4}\ \ \sqrt{18\ \ 2}\ \ \sqrt{16\ \ 25}$ $\quad b)\ \ \sqrt{6\ \ \sqrt{2\ \ \sqrt{2\ \ \sqrt[3]{\ \ 8}}}}$

$c)\ \ (\sqrt[3]{5}\ \ \sqrt[3]{4})\ \ (\sqrt[3]{5^2}\ \ \sqrt[3]{5}\ \ \sqrt[3]{4}\ \ \sqrt[3]{4^2})$ $\quad d)\ \ \sqrt[3]{\sqrt[3]{\dfrac{64}{125}}^{\ 2}\ \ \dfrac{4}{5}}$

$e)\ \ \sqrt{\dfrac{\sqrt{25}\ \ \sqrt[3]{\ \ 27}}{\sqrt[3]{64}\ \ \sqrt[5]{\ \ 32}}}$ $\quad f)\ \ \sqrt{(\ \ \sqrt[5]{\ \ 32})^2\ \ \sqrt{(\sqrt[3]{\ \ 8})^4\ \ (\sqrt[3]{\ \ 27})^2}}$

$g)\ \ (\ \ \sqrt{16}\ \ \sqrt{5}\ \ \sqrt{4}\ \ \sqrt{5})^3$ $\quad h)\ \ \sqrt[3]{\dfrac{\sqrt{16}}{\sqrt[3]{8}}\ \ 3^{\ 2}\ \ (\ \ 3)^2}$

$i)\ \ \sqrt{\sqrt[3]{100}}^{\ \sqrt[3]{27}^{\sqrt{4}}}$ $\quad j)\ \ (1\ \ \sqrt{2})^2\ \ (1\ \ \sqrt{2})^2$

$k)\ \ \sqrt{\dfrac{1}{\sqrt{\dfrac{1}{\sqrt{256}}}}\ \ \sqrt{\sqrt{16}}}$ $\quad l)\ \ (1\ \ \sqrt{2}\ \ \sqrt{3})^2\ \ (1\ \ \sqrt{2}\ \ \sqrt{3})^2$

$m)\ \ \sqrt{\dfrac{\sqrt[3]{\ \ 8}\ \ \sqrt[3]{\ \ 9}\ \ \sqrt[3]{\ \ 8}\ \ \sqrt[3]{\ \ 8}}{\sqrt{16}\ \ \sqrt[5]{\ \ 32}}\ \ \sqrt[3]{\ \ 81}}$ $\quad n)\ \ \sqrt{\dfrac{\sqrt[3]{64}\ \ (\sqrt{3})^{\sqrt{4}}\ \ (\ \ \sqrt{4})^{\sqrt[3]{27}}}{(\sqrt[5]{\ \ 32})^{\sqrt{25}}\ \ (\ \ \sqrt[3]{\ \ 64})^{\sqrt[5]{32}}}}$

Example:

$j)\ \ (1\ \ \sqrt{2})^2\ \ (1\ \ \sqrt{2})^2\ \ (1\ \ \sqrt{2})\ \ (1\ \ \sqrt{2})\ \ (1\ \ \sqrt{2})\ \ (1\ \ \sqrt{2})$
$1\ \ 1\ \ 1\ \ \sqrt{2}\ \ \sqrt{2}\ \ 1\ \ \sqrt{2}\ \ \sqrt{2}\ \ 1\ \ 1\ \ 1\ \ \sqrt{2}\ \ \sqrt{2}\ \ 1\ \ \sqrt{2}\ \ \sqrt{2}\ \ \ 4\ \ \sqrt{2}$

4. Use a calculator to evaluate the following expressions. Round the answer to the nearest thousandth.

$a)\ \ 1\ \ \sqrt{1\ \ \sqrt[3]{2}}$ $\quad b)\ \ \sqrt[3]{\ \ 10}\ \ (2\ \ \sqrt{3})^{\sqrt{2}}$ $\quad c)\ \ \sqrt{(\sqrt[5]{25})^{(1\ \ \sqrt[3]{2})}}$ $\quad d)\ \ 1\ \ \dfrac{\sqrt{3}}{\sqrt[3]{2}}^{\ \frac{1}{2\ \ \sqrt[5]{3}}}$

40. Substitution

40.1 *(Algebraic Expressions)* An algebraic expression contains a well-formed sequence of operations between numbers or/and variables. Examples:

$2 \quad / x \quad y$ *is not well formed*

$2 \quad x \quad y$ *is well formed*

40.2 *(Substitution)* Substitution is the process of replacing the variables of an expressions with numbers in order to evaluate the expression. Example: The value of the following expression:

$x \quad y \quad 2$

if: $x \quad 3$

$y \quad 5$

is: $(\ 3) \quad (\ 5) \quad 2 \quad 8 \quad 2 \quad 10$

40.3 *(Named Expressions or Formulas)* You can give a name to an expression to make it available for further reference. Example:

$a \quad F / m$

$s \quad a \quad t^2 / 2$

If *F = 10*; *m = 5*; *t = 2* then:

$a \quad F / m \quad 10/5 \quad 2$

$s \quad a \quad t^2 / 2 \quad 2 \quad 2^2 / 2 \quad 4$

40.4 *(Function Notation)* You can use an expression to define a function. A function has a name followed (in parentheses) by a list of parameters separated by commas. Example:

$f(x) \quad x^2 \quad 3 \quad x$

$g(x,y) \quad \sqrt{x^2 \quad y^2}$

You can use substitution to find the value of a function for the given values of parameters. Example:

$f(\ 1) \quad (\ 1)^2 \quad 3 \quad (\ 1) \quad 1 \quad 3 \quad 4$

$g(\ 4,\ 3) \quad \sqrt{(\ 4)^2 \quad (\ 3)^2} \quad \sqrt{16 \quad 9} \quad 5$

40.5 *(Order of Operations)* An expression can also contain functions. To find the value of the expression, replace the functions with their values. Example:

$f(x) \quad x^2 \quad x; \quad E \quad 1 \quad f(1) \quad [f(2)]^2 \quad ?$

$f(1) \quad 1^2 \quad 1 \quad 0; \quad f(2) \quad 2^2 \quad 2 \quad 2$

$E \quad 1 \quad 0 \quad 2^2 \quad 3$

1. For each case find whether or not the expression is well-formed:

a) $x \quad 3$ b) $x \quad 5$ c) $x \quad y$ d) x^2 e) $x/ \quad y$

f) $(\ 3\ )x \quad 2$ g) $x^{\ 2} \quad 5$ h) $x^{(y \ 4}$ i) $\frac{x}{2 \quad y \quad z}$ j) $\frac{2 \quad x}{}$

2. If x = -2, find the value of each expression:

a) $x \quad 3 \quad x$ b) $(\ 2)^x$ c) $x \quad (\ 4) \quad x$ d) $\sqrt{2 \quad 3 \quad x}$ e) $x^3 \quad 2 \quad x$

f) $\sqrt[3]{(1 \quad x)^5}$ g) $\frac{4}{x \quad 2}$ h) $\sqrt{\frac{2 \quad x}{x \quad 1}}$ i) $\sqrt{x^x}$ j) $(x \quad 1) \quad (x \quad 2)$

3. If x = -2 and y = +3, find the value of each expression:

a) $x \quad y \quad 2$ b) x^y c) $\sqrt[y]{x^3}$ d) $\sqrt{\ 2 \quad (x \quad 2 \quad y)}$ e) $x^3 \quad y^2$

f) $\left(\frac{y \quad 1}{x \quad 1}\right)^2$ g) y^x h) $\sqrt{\frac{2 \quad x}{3 \quad y}}$ i) $x^2 \quad y^2 \quad (x \quad y)^2$ j) $(x \quad 1) \quad (y \quad 2)$

4. For each case, use substitution to find the value of the named expression:

a) *Given*:
$x \quad 4$
$y \quad 3$
Find:
$z \quad \sqrt{\ \ x} \quad y^2$
$u \quad z \quad 3$
$v \quad \sqrt[3]{u}$

b) *Given*:
$a \quad 2$
$b \quad 4$
$c \quad 3$
Find:
$d \quad b^a \quad c^a$
$e \quad \sqrt[a]{b \quad c \quad 3}$
$f \quad d \quad e^2$

c) *Given*:
$L \quad 2$
$W \quad 5$
$H \quad 3$
Find:
$A \quad 2 \quad (L \quad W \quad L \quad H \quad W \quad H)$
$V \quad L \quad W \quad H$
$D \quad \sqrt{L^2 \quad W^2 \quad H^2}$

5. The functions f, g and h are defined below. Find the values of these functions for each case:

$f(x) \quad x \quad \sqrt[3]{x} \quad x^2; \quad g(x,y) \quad x^2 \quad y^2 \quad x \quad y; \quad h(x,y,z) \quad \frac{x^2 \quad 2 \quad y}{\sqrt{y} \quad x/z}$

a) $f(\ 1)$ b) $g(\ 1,\ 2)$ c) $h(\ 4,4,\ 2)$ d) $g(\ 1,0)$

e) $f(f(1))$ f) $g(f(0), f(1))$ g) $h(g(0,0), f(1),1)$ h) $f(g(3,1) \quad 2)$

6. The functions f(x) and g(x,y) are defined below. Find the value of each expression. One case is solved for you as an example:

$f(x) \quad x \quad x^2; \quad g(x,y) \quad \frac{x \quad 1}{(y \quad 1)^2}$

a) $f(1) \quad f(0)$ b) $g(1,1) \quad g(0,0)$ c) $\frac{f(2)}{g(\ 1,0)}$ d) $2 \quad f(\ 4) \quad g(0,1)$

e) $f(1 \quad f(0))$ f) $g(1 \quad f(0),1 \quad f(1))$ g) $[g(2,1)]^{f(2)}$ h) $1 \quad f(0) \quad g(0,0)$

Example:

g) $[g(2,1)]^{f(2)}; \quad g(2,1) \quad \frac{2 \quad 1}{(1 \quad 1)^2} \quad \frac{1}{4}; \quad f(2) \quad 2 \quad 2^2 \quad 2 \quad 4 \quad 2$

$\left(\frac{1}{4}\right)^{\ 2} \quad (4^{\ 1})^{\ 2} \quad 4^2 \quad 16$

Answers

1. Understanding integers (Page 5)

1. *a*) *yes* *b*) *no* *c*) *yes* *d*) *no* *e*) *yes* *f*) *yes*
   g) *no* *h*) *yes* *i*) *no* *j*) *yes* *k*) *no*

2. *a*) ,3 *b*) ,2 *c*) *undefined*, 0 *d*) ,2 *e*) ,11
   f) ,10 *g*) ,4 *h*) ,3 *i*) ,22 *j*) ,101

3. *a*) 5 *b*) 2 *c*) 1 *d*) 0 *e*) 3

4. *a*) 1 *b*) 20 *c*) 300 *d*) 41 *e*) 55

5. *a*) *positive* *b*) *negative* *c*) *positive* *d*) *negative*
   e) *neither* *f*) *positive* *g*) *negative* *h*) *negative*
   i) *positive* *j*) *negative*

6. *Z* { 1, 2, 3, 4,...}

7. {1,3,5,7,9}

8. *a*) *false* *b*) *false* *c*) *false* *d*) *true* *e*) *false*
   f) *false*

9. *a*) 7 *b*) 11 *c*) 5 *or* 5 *d*) 0 *e*) *not an integer*

10. *a*) *b*) *c*) *d*) *e*) *not defined*
    f) *g*) *h*) *i*) 0 *j*) *not defined*

2. Absolute value, sign, and opposite (Page 6)

1. *a*) 3 *b*) 2 *c*) 2 *d*) 3 *e*) 0 *f*) 10 *g*) 100
   h) 12 *i*) 7 *j*) 7

2. *a*) 0 *b*) 1 *c*) 2 *d*) 3 *e*) 10 *f*) 5 *g*) 123

3. *a*) *b*) *c*) *d*) *not defined* *e*) *f*) *g*)
   h) *i*) *j*)

4. *a*) 12 *b*) 20 *c*) 3 *d*) 3 *e*) 20 *f*) 0 *g*) 11
   h) 2 *i*) 7 *j*) 77

5. *a*) 2 *b*) 5 *c*) 3 *d*) 4 *e*) 20 *f*) 3 *g*) 3
   h) 2 *i*) 2 *j*) 5

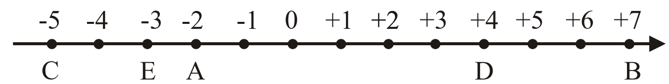
6. *a*) 5 *b*) 3 *c*) 4 *d*) 3 *e*) 7 *f*) 4 *g*) 3
   h) 4 *i*) 15 *j*) 4

7. *a*) 6 *b*) 2 *c*) 6 *d*) 10 *e*) 3 *f*) 8 *g*) 5
   h) 6 *i*) 2 *j*) 4 *k*) 3 *l*) 2 *m*) 5 *n*) 8 0) 3

3. Number line (Page 7)

1. *A*) 2 *B*) 1 *C*) 5 *D*) 4 *E*) 8

2.



3. *A*) 40 *B*) 30 *C*) 20 *D*) 10 *E*) 0

4. *A*) 8 *B*) 5 *C*) 2 *D*) 3 *E*) 7

5. *A*'(6) *B*'(4) *C*'(1) *D*'(3) *E*'(6)

6. *a*) 5 *b*) 5 *c*) 7 *d*) 2 *e*) 2 *or* 8
   f) 8 *or* 4 *g*) 3 *or* 3

4. Comparing integers (Page 8)

1. *a*) 1 2 *b*) 5 3 *c*) 0 5 *d*) 3 3 *e*) 1 0
   f) 0 0

2. *a*) 1 2 *b*) 1 0 2
   c) 3 2 0 1 *d*) 2 1 0 2 3
   e) 3 2 1 1 2 3 *f*) 3 2 1 0 2 3

3. *a*) 1 2 *b*) 1 0 2 *c*) 0 1 2 3
   d) 4 3 2 0 2 *e*) 5 4 2 2 4 5
   f) 3 1 1 2 5

4. *a*) 0 2 *b*) 2 2 *c*) 3 1 *d*) 5 4 *e*) 0 3
   f) 3 1

5. *a*) 1 2 *b*) 1 0 *c*) 1 1 *d*) 0 5 *e*) 0 0
   f) 3 7 *g*) 4 2 *h*) 2 4 *i*) 5 5
   j) 7 5

6. *a*) *true* *b*) *false* *c*) *false* *d*) *false* *e*) *false*
   f) *false* *g*) *false* *h*) *true* *i*) *true* *j*) *false*

7. *a*) *true* *b*) *true* *c*) *true* *d*) *true* *e*) *true*
   f) *true* *g*) *true* *h*) *false* *i*) *true* *j*) *false*

8. *a*) *true* *b*) *true* *c*) *false* *d*) *false* *e*) *false*
   f) *false* *g*) *false* *h*) *true* *i*) *true* *j*) *true*

9. *a*) {2} *b*) { 1,0} *c*) { 2, 1,0} *d*) {0,1} *e*) {2,3,4,5}
   f) { 2} *g*) {0,1,2,3} *h*) {0,1,2,3,4} *i*) { 2, 1}
   j) { 1,0,1,2}

Answers

5. Applications of integers (Page 9)

1. a) 5 b) 4 c) 0 d) 1 e) 2
2. a) 2 b) 2 c) 0 d) 3 e) 5
3. a) 10 b) 15 c) 0 d) 100 e) 273
4. a) 10 b) 5 c) 0 d) 3 e) 10
5. a) 1000 b) 2 c) 20 d) 0 e) 7
6. a) 2 b) 5 c) 7 d) 3 e) 0
7. a) 500 b) 150 c) 0 d) 31 e) 125
8. a) 50 b) 550 c) 40 d) 45 e) 0

6. Addition of integers. Axioms (Page 10)

1. a) 1,2 *addends*; 3 *sum* b) 2, 1 *addends*; 1 *sum* c) 2, 2 *addends*; 4 *sum* d) 7, 3 *addends*; 4 *sum* e) 4, 6 *addends*; 10 *sum* f) 2,4 *addends*; 6 *sum*
2. a) 1 0 b) 3 2 c) 0 5 d) 0 (1) e) 4 (3) f) (5) (4)
3. a) 0 b) 1 c) 5 d) 1 e) 4 f) 5
4. 0 *for all cases*
5. a) 15 b) 5 c) 25 d) 7 e) 33 f) 37 g) 3 h) 3 i) 5
6. a) 2 b) 5 c) 0 d) 3 e) 0 f) 5 g) 9 h) 20 i) 8 j) 27 k) 30 l) 0

7. Addition of integers using the number line (Page 11)

1. a) 2 b) 3 c) 5 d) 1 e) 1 f) 5 g) 3 h) 3 i) 2 j) 4 k) 2 l) 2
2. a) (2) (5) b) (1) (2) c) (4) (4) d) (4) (3) e) 0 (2)
3. a) 0 b) 5 c) 6 d) 2 e) 2 f) 4 g) 4 h) 1 i) 2
4. (1) (5) (3) (2) (5)
5. a) 1 2 b) 3 2 c) 2 5 d) 2 2 3 e) 5 3 1 f) 1 2 4 5
6. a) (1) (2) b) (2) (3) c) (3) (5) d) (1) (2) (3) (4) e) (2) (2) (3) (3) f) (5) (4) (3) (2) (1) (2)
7. a) 3 b) 6 c) 2 d) 6 e) 1 f) 3 g) 2 h) 2 i) 4 j) 5 k) 6 l) 3
8. a) 3 b) 1 c) 3 d) 3

8. Addition of integers using rules (Page 12)

1. a) 4 b) 10 c) 22 d) 54 e) 125 f) 444 g) 235 h) 400 i) 1135 j) 1575 k) 4075 l) 66666 m) 6 n) 65 o) 3210 p) 356
2. a) 3 b) 5 c) 6 d) 225 e) 250 f) 800 g) 4500 h) 5555 i) 1334 j) 9 k) 90 l) 1905
3. a) 2 b) 2 c) 3 d) 2 e) 5 f) 5 g) 35 h) 40 i) 50 j) 100 k) 200 l) 425 m) 1250 n) 50 o) 2222 p) 100
4. a) 1 b) 4 c) 3 d) 5 e) 7 f) 15 g) 5 h) 25 i) 150 j) 25 k) 450 l) 100 m) 1850 n) 5100 o) 2200 p) 225
5. a) 10 b) 2 c) 6 d) 80 e) 111 f) 1000 g) 222 h) 10 i) 300 j) 200 k) 20 l) 3
6. a) 10 b) 0 c) 6 d) 1 e) 90 f) 15

Answers

9. Subtraction of integers (Page 13)

1. *a*) 3 *b*) 4 *c*) 1 *d*) 0 *e*) 6 *f*) 1 *g*) 3
h) 2 *i*) 8 *j*) 3 *k*) 7 *l*) 2

2. *a*) (2) (4) 2 *b*) 0 (4) 4 *c*) (2) (3) 5
d) (1) (2) 3 *e*) (2) (4) 2

3. *a*) 4 *b*) 4 *c*) 2 *d*) 7 *e*) 1 *f*) 2 *g*) 5
h) 4 *i*) 8 *j*) 1 *k*) 5 *l*) 12

4. *a*) 4 *b*) 4 *c*) 5 *d*) 10 *e*) 3 *f*) 6 *g*) 5
h) 20

5. *a*) 5 *b*) 3 *c*) 3 *d*) 2 *e*) 4 *f*) 6 *g*) 5 *h*) 5

6. *a*) 2 *b*) 5 *c*) 2 *d*) 10 *e*) 10 *f*) 8 *g*) 3
h) 2 *i*) 0

7. *a*) 3 *b*) 7 *c*) 2 *d*) 2

10. Order of operations (I) (Page 14)

1. *a*) 0 *b*) 2 *c*) 3 *d*) 2 *e*) 4 *f*) 2 *g*) 0
h) 4 *i*) 5

2. *a*) 0 *b*) 5 *c*) 5 *d*) 4 *e*) 1 *f*) 3 *g*) 3
h) 6 *i*) 50

3. *a*) 3 *b*) 4 *c*) 7 *d*) 8 *e*) 4 *f*) 6 *g*) 8
h) 5 *i*) 1

4. *a*) 2 *b*) 5 *c*) 3 *d*) 5 *e*) 4 *f*) 8 *g*) 4 *h*) 5

5. *a*) 5 *b*) 2 *c*) 6 *d*) 9 *e*) 1 *f*) 1 *g*) 4
h) 9

6. *a*) 4 *b*) 1 *c*) 1 *d*) 6 *e*) 0 *f*) 0

11. Equalities and inequalities (Page 15)

1. *a*) *false* *b*) *false* *c*) *false* *d*) *true* *e*) *false*
f) *false*

2. *a*) *true* *b*) *false* *c*) *false* *d*) *true* *e*) *true* *f*) *true*
g) *true* *h*) *false* *i*) *true* *j*) *true* *k*) *true* *l*) *false*

3. *a*) *true* *b*) *false* *c*) *false* *d*) *true* *e*) *true* *f*) *false*
g) *false* *h*) *true* *i*) *false*

Page 15 - continue

4. *a*) *true* *b*) *false* *c*) *false* *d*) *true* *e*) *true* *f*) *true*

5. *a*) *true* *b*) *true* *c*) *true* *d*) *false* *e*) *true* *f*) *true*
g) *false* *h*) *true*

6. *a*) *false* *b*) *true* *c*) *true* *d*) *true* *e*) *true* *f*) *true*

7. *a*) *true* *b*) *true* *c*) *false* *d*) *true* *e*) *false* *f*) *true*
g) *false* *h*) *false*

12. Equivalent equalities and inequalities (Page 16)

1. *a*) 4 0; *false* *b*) 2 4; *true* *c*) 2 4; *true*
d) 7 3; *true* *e*) 0 0; *true* *f*) 3 4; *true*

2. *The answer may vary.*
a) 2 1 3; 2 3 1 0; 0 1 2 3
b) 2 3 4 0; 0 4 2 3; 2 4 3
c) 4 6 1 0; 0 1 4 6; 4 1 6
d) 5 3 2 1 0; 0 2 1 5 3; 5 1 2 3
e) 2 1 3 3 2 0; 0 3 2 2 1 3; 2 1 3 3 2
f) 10 15 5 15 10; 10 15 10 5 15;
0 5 15 10 10 15

3. *a*) 2 2 4 0 *b*) 3 1 4 0 *c*) 1 2 0
d) 3 2 2 0 *e*) 1 2 2 3 0
f) 10 15 5 10 15 0

4. *a*) 0 3 4 7 *b*) 0 2 1 3 *c*) 0 1 2 1 2
d) 0 3 1 1 *e*) 0 10 15 5 10
f) 0 3 2 3 7 5

5. *a*) 1 1 2 *b*) 1 2 *c*) 2 3 *d*) 4 3
e) 1 4 5 2 2 *f*) 1 2 3 2 3 4

6. *a*) 2 1 1 *b*) 0 1 *c*) 3 2 *d*) 0 1 *e*) 2 1
f) 1 3 5

7. *a*) 0 0 *b*) 0 1 *c*) 0 1 *d*) 3 0 *e*) 1 0
f) 0 1

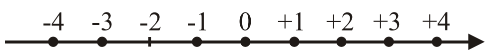
8. *a*) *true* *b*) *true* *c*) *true* *d*) *false* *e*) *false*
f) *true* *g*) *true* *h*) *true*

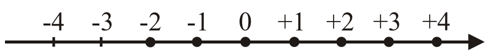
Answers

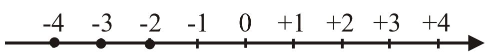
13. Equations (Page 17)

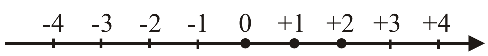
1. *a*) yes *b*) no *c*) no *d*) yes *e*) no *f*) yes
2. *a*) 1 *b*) 2 *c*) 6 *d*) 2 *e*) 1 *f*) 0 *g*) 3 *h*) 1 *i*) 2
3. *a*) 1 *b*) 5 *c*) 10 *d*) 0 *e*) 7 *f*) 5 *g*) 5 *h*) 0 *i*) 10
4. *a*) 0 *b*) 1 *c*) 6 *d*) 2 *e*) 1 *f*) 25
5. *a*) 4 *b*) 0 *c*) 3 *d*) 6 *e*) 4 *f*) 3
6. *a*) 1 *b*) 0 *c*) 2 *d*) 5 *e*) 5 *f*) 5 *g*) 1 *h*) 1 *i*) 2

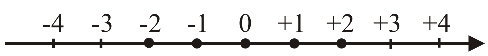
14. Inequations (Page 18)

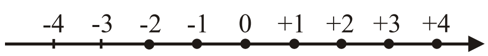
1. *a*) yes *b*) yes *c*) yes *d*) yes *e*) no *f*) yes
2. *a*)


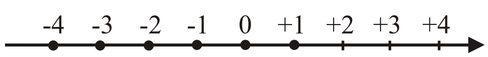
b)


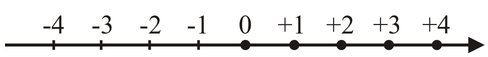
c)


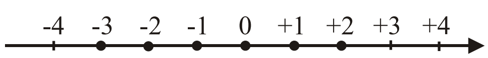
d)


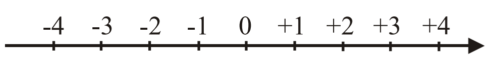
e)


f)


g)


h)


i)


j)


3. *a*) *x* 2 *b*) *x* 1 *c*) *x* 1 *d*) *x* 1 *e*) *x* 5 *f*) *x* 15
4. *a*) *x* 1 *b*) *x* 1 *c*) *x* 5 *d*) *x* 0 *e*) *x* 7 *f*) *x* 5 *g*) *x* 0 *i*) *x* 3
5. *a*) *x* 4 *b*) *x* 2 *c*) *x* 2 *d*) *x* 2 *e*) *x* 1 *f*) *x* 5
6. *a*) *x* 3 *b*) *x* 8 *c*) *x* 3 *d*) *x* 6 *e*) *x* 6 *f*) *x* 0
7. *a*) *x* 10 *b*) *x* 7 *c*) *x* 5 *d*) *x* 8 *e*) *x* 4 *f*) *x* 15
8. *a*) *x* 8 *b*) *x* 7 *c*) *x* 2 *d*) *x* 6 *e*) *x* 3 *f*) *x* 2

15. Multiplication of integers (I) (Page 19)

1. *a*) 3 1 *b*) 3 5 *c*) 5 (3) *d*) 3 (2) *e*) 6 (1) *f*) 7 (7)
2. *a*) 2 2 2 *b*) 5 5 5 5 *c*) 1 1 *d*) 5 5 5 5 5 *e*) 0 0 0 *f*) 3 3 3 3 3 *g*) 4 4 4 *h*) 10 10 10 10 10 *i*) 1 *j*) 18 18 18
3. *a*) 5 1 1 1 1 1 *b*) 3 3 2 2 2 *c*) 4 4 2 2 2 2 *d*) 6 6 2 2 2 2 2 2 *e*) 5 5 5 3 3 3 3 3 *f*) 5 5 2 2 2 2 2 *g*) 3 3 3 3 4 4 4 *h*) 3 3 3 3 3 3 *i*) 1 1 *j*) 0 0
4. *a*) 1 *b*) 10 *c*) 15 *d*) 30 *e*) 6 *f*) 100 *g*) 50 *h*) 30
5. *a*) 6 *b*) 5 *c*) 3 *d*) 4 *e*) 12 *f*) 1 *g*) 9 *h*) 12 *i*) 20 *j*) 30
6. *a*) 1 *b*) 6 *c*) 3 *d*) 10 *e*) 8 *f*) 20 *g*) 12 *h*) 9 *i*) 20 *j*) 25
7. *a*) 0 *b*) 0 *c*) 0 *d*) 1 *e*) 1 *f*) 10 *g*) 0 *h*) 10
8. *a*) 20 *b*) 20 *c*) 20 *d*) 20 *e*) 100 *f*) 100 *g*) 100 *h*) 100 *i*) 100 *j*) 60 *k*) 200 *l*) 125 *m*) 36 *n*) 120 *o*) 300 *p*) 300 *q*) 123 *r*) 531 *s*) 0 *t*) 111
9. *a*) 6 *b*) 8 *c*) 0 *d*) 333 *e*) 2 *f*) 9 *g*) 0 *h*) 0

16. Multiplication of integers (II) (Page 20)

1. *a*) 20 4 (5) *b*) 16 (4) (4) *c*) 10 (5) (2) *d*) (3) (4) 12 *e*) 1 1 (1) *f*) 1 (0) 0 *g*) 100 (10) (10) 12 *h*) 60 (15) (4) *i*) 36 (12) (3)
2. *a*) 30 *b*) 140 *c*) 90 *d*) 240 *e*) 1100 *f*) 1500 *g*) 3600 *h*) 3000 *i*) 9000
3. *a*) 14 *b*) 9 *c*) 16 *d*) 14 *e*) 21 *f*) 25 *g*) 7 *h*) 4 *i*) 24
4. *a*) 1 (2 3) 5 *b*) 2 (3 5) 4 *c*) 4 (3 5) 32 *d*) 5 (1 5) 20 *e*) 10 (3 5) 20 *f*) 7 (1 5 2) 14 *g*) 2 (1 3 2) 4 *h*) 2 (1 3 4) 4 *i*) 11 (1 5 2 3) 55 *j*) 10 (2 3 4) 30 *k*) 15 (2 3) 75 *l*) 25 (1 2 3 4) 50

Answers

5. *a*) 5 *b*) 1 *c*) 16 *d*) 2 *e*) 4 *f*) 8 *g*) 12
h) 6 *i*) 0

6. *a*) 297 *b*) 195 *c*) 612 *d*) 204 *e*) 399 *f*) 969

17. Order of operations (II) (Page 21)

1. *u* *unary*; *b* *binary*
a) *u*; *b*; *u*; *b*; *u*; *b*; *u* *b*) *u*; *u*; *u*; *b*; *b*
c) *b*; *u*; *b*; *u*; *b*; *u*; *b* *d*) *u*; *b*; *b*; *b*

2. *a*) 6 *b*) 6 *c*) 6 *d*) 6 *e*) 12 *f*) 16
g) 0 *h*) 0 *i*) 0 *j*) 120 *k*) 96 *l*) 1200

3. *a*) 7 *b*) 5 *c*) 1 *d*) 7 *e*) 1 *f*) 5
g) 10 *h*) 23 *i*) 1 *j*) 2 *k*) 3 *l*) 11
m) 23 *n*) 26 *o*) 0 *p*) 6 *q*) 5 *r*) 20

4. *a*) 9 *b*) 5 *c*) 1 *d*) 3 *e*) 5 *f*) 1
g) 4 *h*) 12 *i*) 5 *j*) 1 *k*) 4 *l*) 5
m) 21 *n*) 3 *o*) 2 *p*) 64 *q*) 8 *r*) 0 *s*) 5

5. *a*) 2 *b*) 1 *c*) 2 *d*) 4 *e*) 16 *f*) 29

18. Division of integers (I) (Page 22)

1. *a*) $\frac{2}{1}$ 1 *b*) $\frac{9}{3}$ 3 *c*) $\frac{8}{2}$ 4 *d*) $\frac{1}{1}$ 1
e) $\frac{10}{5}$ 2 *f*) $\frac{15}{3}$ 5 *g*) $\frac{0}{1}$ 0 *h*) *not possible*

2. *a*) 2 1 (1) *b*) 12 (3) (4) *c*) 0 1 0
d) 10 (1) (10) *e*) 20 (4) (5) *f*) 8 (4) (2)
g) 10 5 (2) *h*) 6 2 (3) *i*) 0 0 (2)
j) 100 10 (10)

3. *a*) 1 *b*) 2 *c*) 0 *d*) 1 *e*) 4 *f*) 3
g) 3 *h*) 4 *i*) 3 *j*) 3 *k*) 2 *l*) 5

4. *a*) 3 *b*) 3 *c*) 2 *d*) 1 *e*) 3 *f*) 2
g) 4 *h*) 5

5. *a*) 1 *b*) 3 *c*) 1 *d*) 1 *e*) 1 *f*) 1 *g*) 1 *h*) 5
i) *not defined* *j*) 6 *k*) 0 *l*) *not defined*

6. *a*) 4 *b*) 3 *c*) 9 *d*) 5 *e*) 3 *f*) 2
g) 11 *h*) 5 *i*) 5 *j*) 4 *k*) 8 *l*) 25

7. *a*) 1 *b*) 1 *c*) 3 *d*) 3 *e*) 6
f) 5 *g*) 2 *h*) 0 *i*) 0 *j*) 3

19. Division of Integers (I) (Page 23)

1. *a*) $\frac{3}{4}$ *b*) $\frac{7}{2}$ *c*) $\frac{8}{3}$ *d*) $\frac{3}{2}$ *e*) $\frac{4}{3}$ *f*) $\frac{16}{9}$
g) $\frac{4}{3}$ *h*) $\frac{5}{3}$ *i*) $\frac{5}{2}$ *j*) $\frac{5}{2}$ *k*) $\frac{3}{2}$ *l*) $\frac{9}{8}$

2. *a*) 0.25 *b*) 0.4 *c*) 0.25 *d*) 0.625 *e*) 0.7
f) 0.2 *g*) 0.2 *h*) 0.4 *i*) 0.25 *j*) 3.5
k) 1.5 *l*) 0.25

3. *a*) $0.\overline{3}$ *b*) $0.\overline{6}$ *c*) $0.\overline{3}$ *d*) $1.\overline{3}$ *e*) $0.1\overline{6}$
f) $0.\overline{09}$ *g*) $0.\overline{3}$ *h*) $0.\overline{3}$ *i*) $0.\overline{3}$ *j*) $0.1\overline{6}$

4. *a*) 2, 4, 6, 8, 10 *b*) 5, 10, 15, 20, 25
c) 6, 12, 18, 24, 30 *d*) 1, 2, 3, 4, 5
e) 7, 14, 21, 28, 35 *f*) 2, 4, 6, 8, 10
g) 5, 10, 15, 20, 25 *h*) 4, 8, 12, 16, 20
i) 8, 16, 24, 32, 40 *j*) 10, 20, 30, 40, 50

5. *a*) 7 2 3 1 *b*) 9 4 2 1 *c*) 9 3 3 0
d) 11 2 4 3 *e*) 10 3 3 1 *f*) 17 3 5 2
g) 19 3 5 4 *h*) 21 2 10 1

6. *a*) 2*R*1 *b*) 1*R*3 *c*) 1*R*4 *d*) 3*R*2 *e*) 2*R*1 *f*) 3*R*3
g) 5*R*0 *h*) 7*R*1

7. *a*) 9 2 4 1 *b*) 9 (4) (2) 1 *c*) 9 4 (3) 3
d) 9 (4) 3 3 *e*) 17 5 3 2
f) 17 (5) (3) 2 *g*) 17 5 (4) 3
h) 17 (5) 4 3

8. *a*) 2*R*1 *b*) 2*R*1 *c*) 3*R*2 *d*) 3*R*2 *e*) 3*R*3 *f*) 3*R*3
g) 4*R*1 *h*) 4*R*1

20. Order of operations (III) (Page 24)

1. *a*) 3 *b*) 2 *c*) 5 *d*) 2 *e*) 8 *f*) 8 *g*) 4
h) 5 *i*) 3

2. *a*) 6 *b*) 4 *c*) 2 *d*) 18 *e*) 8 *f*) 5 *g*) 3
h) 9 *i*) 6

3. *a*) 4 *b*) 1 *c*) 3 *d*) 2 *e*) 6 *f*) 3 *g*) 1
h) 3 *i*) 5

4. *a*) 9 *b*) 9 *c*) 9 *d*) 3 *e*) 4 *f*) 12 *g*) 5
h) 2 *i*) 1

5. *a*) 2 *b*) 4 *c*) 2 *d*) 3 *e*) 3 *f*) 3 *g*) 4
h) 7 *i*) 1 *j*) 0 *k*) 2 *l*) 2 *m*) 2

Answers

21. Equalities and equations (Page 25)

1. *a)* (3) (1) (3) (2 3); *true*
   b) (2) (2) [2 (3) 4] (2); *true*
   c) (1) 4 (1) [(5 10) (5) 3]; *true*
   d) [4 (3)] (6) [5 (3) 3] (6); *true*

2. *a)* $x \ \frac{15}{3} \ 5$ *b)* $x \ \frac{8}{4} \ 2$ *c)* $x \ \frac{12}{4} \ 3$
   d) $x \ \frac{14}{7} \ 2$ *e)* $x \ \frac{10}{2} \ 5$ *f)* $x \ \frac{9}{3} \ 3$
   g) $x \ \frac{20}{5} \ 4$ *h)* $x \ \frac{0}{3} \ 0$

3. *a)* 4 *b)* 6 *c)* 5 *d)* 4 *e)* 7 *f)* 6 *g)* 5 *h)* 0

4. *a)* 12 *b)* 20 *c)* 16 *d)* 0 *e)* 30 *f)* 7
   g) 21 *h)* 1

5. *a)* 3 *b)* 12 *c)* 12 *d)* 0 *e)* 6 *f)* 12 *g)* 10
   h) 9

6. *a)* 5 *b)* 5 *c)* 5 *d)* 4 *e)* 6 *f)* 3
   g) 10 *h)* 4 *i)* 1 *j)* 4 *k)* 5 *l)* 4

7. *a)* 1 *b)* 5 *c)* 5 *d)* 6 *e)* 8 *f)* 3
   g) 3 *h)* 5 *i)* 5 *j)* 4 *k)* 100 *l)* 13

22. Proportions and equations (Page 26)

1. *a)* 1 *b)* 10 *c)* 2 *d)* 0 *e)* 5 *f)* 2 *g)* 16 *h)* 1

2. *a)* 1 *b)* 10 *c)* 4 *d)* 0 *e)* 2 *f)* 12 *g)* 25 *h)* 2

3. *a)* 3 *b)* 3 *c)* 2 *d)* 8 *e)* 2 *f)* 15 *g)* 7
   h) 3 *i)* 0 *j)* 9 *k)* 6 *l)* 0 *m)* 0 *n)* 4 *o)* 3
   p) 4 *q)* 1 *r)* 1

4. *a)* 5 *b)* 4 *c)* 3 *d)* 3 *e)* 4 *f)* 21
   g) 9 *h)* 2

23. Inequalities and inequations (Page 27)

1. *a)* 2 (1) 2 (2 3); *true*
   b) (6) 3 [2 (1) 5] 3; *true*
   c) 3 0 3 [12 (4) 3]; *true*
   d) [4 (3) (6)] 2 4 2; *true*

2. *a)* (2) (2) (2) (3 4); *true*
   b) (5) (5) [2 (10) 15] (5); *true*
   c) (3) 0 (3) [10 (5) 1]; *true*
   d) [2 (3) (3)] (2) (2) (2); *true*

Page 27 - continue

3. *a)* x 3 *b)* x 5 *c)* x 2 *d)* x 2 *e)* x 2
   f) x 10 *g)* x 10 *h)* x 0 *i)* x 1 *j)* x 0
   k) x 5 *l)* x 10

4. *a)* x 0 *b)* x 0 *c)* x 1 *or* x 0 *d)* 0 x 5
   e) 3 x 0 *f)* 2 x 0 *g)* 0 x 3 *h)* 0 x 2
   i) 0 x 1 *j)* 0 x 3 *k)* x 7 *or* x 0 *l)* 0 x 4

5. *a)* x 6 *b)* x 3 *c)* x 4 *d)* x 3 *e)* x 5
   f) x 25 *g)* 3 x 7 *h)* x 1 *i)* x 3

6. *a)* x 2 *b)* x 1 *c)* x 5 *d)* x 0 *or* x 4
   e) 2 x 2 *f)* x 1 *g)* x 5 *or* x 5
   h) any number *i) no solution* *j)* 4 x 4
   k) 5 x 7 *l)* 4 x 8

24. Powers (Page 28)

1. *a)* 3^7 *b)* $(\ 5)^4$ *c)* $(\ 4)^1$ *d)* $(\ 10)^5$ *e)* $(\ 2)^3$ *f)* $(\ 1)^6$

2. *a)* 3 3 3 3 *b)* (2) (2) (2) *c)* (1) (1)
   d) (3) (3) (3) *e)* (2) (2) (2) (2)
   f) (8) (8) *g)* 10 10 10 *h)* (10) (10)
   i) (5) (5) (5) *j)* 1 1 1 1 1 *k)* (7) (7)
   l) (6) (6) (6)

3. *a)* $\frac{1}{4\ \ 4}$ *b)* $\frac{1}{2}$ *c)* $\frac{1}{(\ 1)\ (\ 1)\ (\ 1)}$ *d)* $\frac{1}{5\ \ 5\ \ 5}$
   e) $\frac{1}{(\ 3)\ (\ 3)}$ *f)* $\frac{1}{10\ \ 10}$ *g)* $\frac{1}{(\ 10)\ (\ 10)\ (\ 10)}$
   h) $\frac{1}{(\ 4)\ (\ 4)}$ *i)* $\frac{1}{1\ 1\ 1\ 1\ 1}$ *j)* $\frac{1}{(\ 5)\ (\ 5)\ (\ 5)\ (\ 5)}$

4. *a)* 8 *b)* $\frac{1}{8}$ *c)* 1 *d)* 1 *e)* 100 *f)* 5 *g)* 100
   h) 1000 *i)* $\frac{1}{100}$ *j)* $\frac{1}{1000}$ *k)* $\frac{1}{100}$ *l)* 1 *m)* 0
   n) not defined *o)* 4 *p)* $\frac{1}{8}$ *r)* $\frac{1}{25}$ *s)* 1

5. *a)* 3^2 *b)* 2^(1) *c)* (2)^3 *d)* (5)^(4)
   e) 10^(2) *f)* (10)^3

6. *a)* 5^2 *b)* $5^{\ 2}$ *c)* $(\ 5)^2$ *d)* $(\ 5)^{\ 2}$ *e)* $(\ 10)^{\ 1}$

7. *a)* 9 *b)* $\frac{1}{100}$ *c)* 25 *d)* $\frac{1}{1000}$ *e)* $\frac{1}{8}$

8. *a)* $\frac{1}{4}$ *b)* $\frac{1}{10}$ *c)* 1000 *d)* 25 *e)* $\frac{1}{4}$

Answers

25. Exponents rules (Page 29)

1. *a*) 2^5 *b*) $(\ 3)^7$ *c*) $(\ 10)^4$ *d*) 5^6 *e*) 0^5
2. *a*) $4^3\ \ 4^2$ *b*) $(\ 2)^1\ \ (\ 2)^2$ *c*) $(\ 10)^2\ \ (\ 10)^5$ *d*) $4^3\ \ 4^1$ *e*) $0^2\ \ 0^2$ *f*) $(\ 1)^3\ \ (\ 1)^1$
3. *a*) 2^3 *b*) 3^3 *c*) $(\ 10)^2$ *d*) $(\ 3)^0$ *e*) $(\ 1)^2$ *f*) $(\ 4)^{\ 4}$ *e*) 3^0
4. *a*) $\frac{5^2}{5^3}$ *b*) $\frac{(\ 5)^1}{(\ 5)^4}$ *c*) $\frac{(\ 10)^6}{(\ 10)^3}$ *d*) $\frac{1^3}{1^5}$ *e*) $\frac{4^1}{4^2}$ *f*) $\frac{(\ 1)^2}{(\ 1)^1}$
5. *a*) 2^6 *b*) $2^{\ 6}$ *c*) 2^6 *d*) $(\ 2)^6$ *e*) 3^1 *f*) $(\ 7)^2$
6. *a*) $2^4\ \ 3^4$ *b*) $(\ 4)^3\ \ 1^3$ *c*) $(\ 5)^2\ \ (\ 2)^2$ *d*) $2^3\ \ (\ 3)^3$ *e*) $(\ 2)^2\ \ 0^2$
7. *a*) 2^4 *b*) $(\ 1)^7$ *c*) $(\ 10)^7$ *d*) 0^2 *e*) $(\ 5)^3$
8. *a*) $\frac{2^4}{3^4}$ *b*) $\frac{(\ 2)^2}{3^2}$ *c*) $\frac{(\ 1)^5}{(\ 2)^5}$ *d*) $\frac{0^3}{3^3}$ *e*) $\frac{(\ 2)^2}{(\ 2)^2}$ *f*) $\frac{5^3}{(\ 3)^3}$
9. *a*) $\frac{1}{2}^{\,3}$ *b*) $\ \frac{1}{2}^{\,5}$ *c*) $\frac{1}{4}^{\,2}$ *d*) 3^7 *e*) 0^4 *f*) $\ \frac{2}{3}^{\,5}$
10. *a*) 2 *b*) 9 *c*) 10000 *d*) $\frac{1}{5}$ *e*) $\frac{1}{5}$ *f*) 100 *g*) 144 *h*) $\frac{1}{4}$ *i*) 2 *j*) 16 *k*) $\frac{1}{16}$

26. Order of operations (Page 30)

1. *a*) 4 *b*) 4 *c*) $\frac{1}{16}$ *d*) $\frac{1}{16}$ *e*) 1 *f*) 1 *g*) $\frac{1}{2}$ *h*) $\frac{1}{4}$ *i*) 1 *j*) $\frac{1}{2}$ *k*) 10 *l*) $\frac{1}{10}$
2. *a*) 512 *b*) 64 *c*) 512 *d*) 512 *e*) 512 *f*) $\frac{1}{512}$ *g*) 512 *h*) $\frac{1}{512}$ *i*) 512 *j*) 65536 *k*) 256 *l*) 256
3. *a*) 7 *b*) 2 *c*) 4 *d*) 1/16 *e*) 16 *f*) 1/2
4. *a*) 6 *b*) 16 *c*) 1 *d*) 7 *e*) 5 *f*) 13
5. *a*) 3 *b*) 0 *c*) 0 *d*) 9 *e*) 4 *f*) 1 *g*) 8 *h*) 4 *i*) 30 *j*) 6 *k*) 0 *l*) 3
6. *a*) 3 *b*) 2 *c*) 4 *d*) 6 *e*) 1 *f*) 2 *g*) 2 *h*) 2 *i*) 17 *j*) 1

27. Divisors (Page 31)

1. *a*) *true* *b*) *false* *c*) *true* *d*) *true* *e*) *false* *f*) *true* *g*) *false* *h*) *false*
2. *a*) { 1, 10} *b*) { 1, 8} *c*) { 1, 5} *d*) { 1, 20} *e*) { 1, 11} *f*) { 1, 3}
3. *a*) { 2, 3, 4, 6} *b*) { 2, 3} *c*) { 3, 5} *d*) { 2, 5} *e*) { 2, 3} *f*) { 2, 4, 8}
4. *a*) { 1, 3, 9} *b*) { 1, 2, 3, 6, 9, 18} *c*) { 1, 2, 3, 4, 6, 8, 12, 24} *d*) { 1, 2, 4, 5, 10, 20} *e*) { 1, 2, 3, 5, 6, 10, 15, 30} *f*) { 1, 3, 7, 21}
5. {2,3,5,7,11,13,17,19,23,29}
6. {32,33,34,35,36,38,39,40,42,44,45,46,48,49}
7. *a*) *prime* *b*) *composite* *c*) *prime* *d*) *prime* *e*) *composite* *f*) *prime* *g*) *composite* *h*) *neither*
8. *B* {2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67, 71,73,79,83,89,91,97}

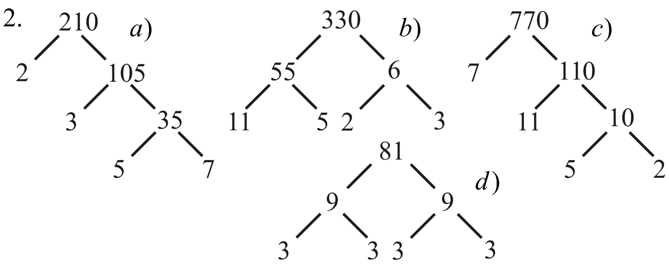
28. Divisibility rules (Page 32)

1. *a*) *no* *b*) *yes* *c*) *yes* *d*) *no* *e*) *yes* *f*) *no* *g*) *yes*
2. *a*) *no* *b*) *yes* *c*) *no* *d*) *yes* *e*) *yes* *f*) *no* *g*) *yes*
3. *a*) *yes* *b*) *yes* *c*) *no* *d*) *no* *e*) *no* *f*) *yes* *g*) *no*
4. *a*) *yes* *b*) *yes* *c*) *yes* *d*) *yes* *e*) *yes* *f*) *no* *g*) *no*
5. *a*) *yes* *b*) *no* *c*) *yes* *d*) *yes* *e*) *no* *f*) *yes* *g*) *yes*
6. *a*) *yes* *b*) *no* *c*) *no* *d*) *yes* *e*) *no* *f*) *no* *g*) *yes*
7. *a*) *yes* *b*) *no* *c*) *yes* *d*) *no* *e*) *no* *f*) *yes* *g*) *no*
8. *a*) *no* *b*) *yes* *c*) *yes* *d*) *no* *e*) *yes* *f*) *yes* *g*) *yes*
9. *a*) *yes* *b*) *no* *c*) *yes* *d*) *yes* *e*) *no*
10. *a*) *yes* *b*) *yes* *c*) *yes* *d*) *yes* *e*) *yes* *f*) *no* *g*) *no*
11. *a*) *yes* *b*) *yes* *c*) *yes* *d*) *no* *e*) *no* *f*) *no* *g*) *no*
12. *a*) *yes* *b*) *yes* *c*) *yes* *d*) *yes* *e*) *no* *f*) *no* *g*) *yes*
13. *a*) *yes* *b*) *no* *c*) *no* *d*) *no* *e*) *yes*

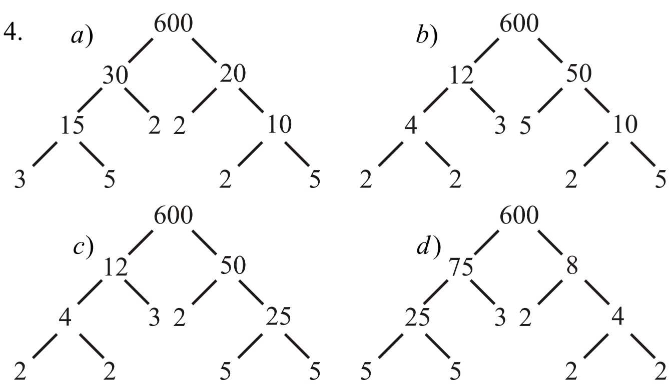
Answers

29. Prime factorization (Page 33)

1. *a*) 210 2^1 3^1 5^1 7^1 *b*) 225 3^2 5^2
c) 150 2^1 3^1 5^2 *d*) 330 2^1 3^1 5^1 11^1

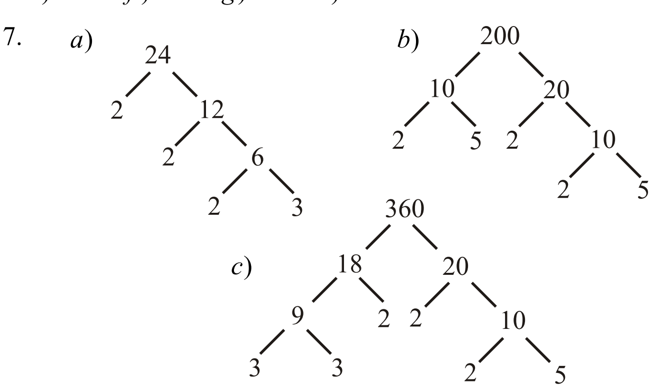
2.


3. *a*) 40 2^3 5^1 *b*) 64 2^6 *c*) 80 2^4 5^1
d) 100 2^2 5^2 *e*) 250 2^1 5^3 *f*) 1024 2^{10}
g) 350 2^1 5^2 7^1 *h*) 83 83^1

4.


5. *a*) 12 2^2 3^1 *b*) 88 2^3 11^1
c) 300 2^2 3^1 5^2 *d*) 4096 2^{12}
e) 17 17^1 *f*) 12345 3^1 5^1 823^1

6. *a*) 72 *b*) 525 *c*) 400 *d*) 900
e) 256 *f*) 243 *g*) 800 *h*) 2310

7.


8. *a*) 2^4 3^6 5^5 7^2 *b*) 3^3 5^4 7^3 *c*) 2^6 3^4 5^4
d) 2^2 3^5 5^2 7^1 *e*) 2^6 3^7 5^8 7^3 *f*) 2^1 3^8 5^2 7^2

30. Set of divisors (Page 34)

1. *a*) {2,3,5,7} *b*) { 1, 2, 5, 10} *c*) {0,1,2,3,4}
d) {3,6,9,12,15} *e*) {11,13,15,17,19}

2. *a*) {1,2,4} *b*) {2,3,4,6,9} *c*) {6,8,15,20} *d*) {1,2,4,8}
e) {5} *f*) {3,4,6,8} *g*) { 4, 1,2} *h*) {1,2,3,4,6}

3. *a*) {1,2} *b*) {1,2,4} *c*) {1,2,4,8} *d*) {1,2,4,16}
e) {1,3,9} *f*) {1,3,9,27}
g) {1,2,4,8,16,32,64,128,256,512,1024} *h*) {1,5,25,125}

4. *a*) {1,2,3,6} *b*) {1,2,5,10} *c*) {1,2,4,8,16} *d*) {1,2,4,5,10,20}
e) {1,2,3,4,6,9,12,18,36} *f*) {1,2,4,5,10,20,25,50,100}
g) {1,5,25,125,625} *h*) {1,2,4,8,16,32,64,128,512,1024,2048}

5. *a*) { 1, 2, 4, 8} *b*) { 1, 2, 4, 5, 8, 10, 20, 40}
c) { 1, 2, 4, 5, 8, 10, 16, 20, 32, 40, 80, 160}
d) { 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200}
e) { 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120,180, 360}
f) { 1, 2, 4, 5, 8, 10,20, 25, 40, 50, 100, 125, 200, 250, 500, 1000}
g) { 1, 2, 4, 8, 16, 32, 64, 128, 256,512}

6. *a*) 6 *b*) 8 *c*) 7 *d*) 12 *e*) 16 *f*) 40 *g*) 15 *h*) 22

7. *a*) {2,3,4} *b*) { 1, 2,0,1} *c*) {4,10,25} *d*) {1,2,4,8}
e) {1,2,3,4,6}

31. Highest Common Factor (HCF) (Page 35)

1. *a*) 4 *b*) 4 *c*) 8 *d*) 9 *e*) 50 *f*) 6

2. *a*) 4 *b*) 2 *c*) 8 *d*) 5 *e*) 8 *f*) 35

3. *a*) 6 *b*) 6 *c*) 1 *d*) 4 *e*) 75

4. *a*) 50 *b*) 6 *c*) 1 *d*) 25

5. *a*) 12 (2 3) *b*) 16 (4 5) *c*) 25 (5 9)
d) 10 (24 10 35) *e*) 6 (2 5 3)
f) 50 (4 5 10) *g*) 64 (1 4) *h*) 6 (10 12 15)

6. *a*) 1 *b*) 1 *c*) 4 *d*) 5 *e*) 1 *f*) $\frac{3}{2}$ *g*) 1 *h*) $\frac{1}{2}$

7. *a*) 1 *b*) $\frac{8}{15}$ *c*) 4 *d*) $\frac{15}{7}$ *e*) 1 *f*) 2 *g*) $\frac{2}{5}$ *h*) $\frac{4}{3}$

Answers

32. Least Common Multiple (LCM) (Page 36)

1. *a*) 30 *b*) 120 *c*) 60 *d*) 60 *e*) 175 *f*) 60
2. *a*) 20 *b*) 90 *c*) 80 *d*) 120 *e*) 150 *f*) 96
3. *a*) 120 2 60 *b*) 150 5 30 *c*) 360 6 60 *d*) 48 2 24 *e*) 250 5 50 *f*) 224 2 112
4. *a*) 24 *b*) 300 *c*) 48 *d*) 180 *e*) 100
5. *a*) 2 *b*) 48 *c*) 20 *d*) $\frac{4}{15}$ *e*) 1 *f*) 20 *g*) $\frac{1}{8}$ *h*) 5
6. *a*) $\frac{5}{6}$ *b*) $\frac{1}{2}$ *c*) $\frac{19}{12}$ *d*) $\frac{1}{30}$ *e*) $\frac{1}{48}$ *f*) $\frac{11}{40}$
7. *a*) $\frac{1}{6}$ *b*) $\frac{3}{2}$ *c*) $\frac{13}{12}$ *d*) $\frac{1}{30}$ *e*) $\frac{23}{48}$

33. Square roots (Page 37)

1. *a*) 0 *b*) 1 *c*) 4 *d*) 9 *e*) 16 *f*) 25 *g*) 36 *h*) 49 *i*) 64 *j*) 81 *k*) 100 *l*) 121 *m*) 144 *n*) 225 *o*) 400 *p*) 2500 *q*) 10000 *r*) 40000
2. *a*) 1 *b*) 2 *c*) 0 *d*) 5 *e*) 9 *f*) 6 *g*) 3 *h*) 8 *i*) 11 *j*) 20 *k*) 10 *l*) 100 *m*) 50 *n*) 200 *o*) 7 *p*) 4 *q*) 12 *r*) 4
3. *a*) 1 *b*) 3 *c*) 0 *d*) 7 *e*) 5 *f*) 9 *g*) 10 *h*) 13 *i*) 20 *j*) 100 *k*) 123 *l*) 99999
4. *a*) 5 *b*) 1 *c*) 7 *d*) 8 *e*) 3 *f*) 11 *g*) 10 *h*) 30 *i*) 100 *j*) 125
5. *a*) 6 *b*) 45 *c*) 54 *d*) 270 *e*) 175 *f*) 75 *g*) 180 *h*) 12500
6. *a*) 33 *b*) 54 *c*) 50 *d*) 125 *e*) 84 *f*) 450
7. *a*) 1.41 *b*) 1.73 *c*) 2.65 *d*) 3.16 *e*) 7.07 *f*) 35.13 *g*) 316.23
8. *a*) 1.414 *b*) 1.732 *c*) 3.162 *d*) 5.477 *e*) 14.142 *f*) 27.875 *g*) 111.108

34. Cubic roots (Page 38)

1. *a*) 0 *b*) 1 *c*) 8 *d*) 27 *e*) 64 *f*) 125 *g*) 216 *h*) 1000 *i*) 8000 *j*) 125000 *k*) 1000000 *l*) 8000000 *m*) 1331 *n*) 1728
2. *a*) 0 *b*) 12 *c*) 1 *d*) 6 *e*) 2 *f*) 4 *g*) 5 *h*) 10 *i*) 3 *j*) 20 *k*) 50 *l*) 11
3. *a*) 1 *b*) 8 *c*) 27 *d*) 64 *e*) 125 *f*) 216 *g*) 1000 *h*) 8000 *i*) 64000 *j*) 1000000 *k*) 27000000 *l*) 1728
4. *a*) 1 *b*) 6 *c*) 2 *d*) 6 *e*) 3 *f*) 10 *g*) 4 *h*) 20 *i*) 5 *j*) 12
5. *a*) 1 *b*) 3 *c*) 5 *d*) 10 *e*) 12 *f*) 12345
6. *a*) 1 *b*) 2 *c*) 4 *d*) 10 *e*) 123
7. *a*) 6 *b*) 50 *c*) 300 *d*) 360 *e*) 100 *f*) 100 *g*) 90 *h*) 40
8. *a*) 18 *b*) 45 *c*) 24 *d*) 32 *e*) 120
9. *a*) 1.913 *b*) 2.714 *c*) 4.309 *d*) 4.642 *e*) 10.726 *f*) 46.416

35. Roots of superior order (Page 39)

1. *a*) 0 *b*) 16 *c*) 81 *d*) 1024 *e*) 625 *f*) 1000000 *g*) 100000000 *h*) 1 *i*) 1 *j*) 16 *k*) 32 *l*) 81 *m*) 10000 *n*) 100000
2. *a*) 0 *b*) 1 *c*) 2 *d*) 5 *e*) 10 *f*) 3 *g*) 1 *h*) 2 *i*) 10 *j*) *not defined* *k*) 3 *l*) 2
3. *a*) 1 *b*) 4 *c*) 2 *d*) 10 *e*) 12 *f*) 54321
4. *a*) 1 *b*) 1 *c*) 4 *d*) 10 *e*) 123
5. *a*) 18 *b*) 250 *c*) 300 *d*) 108 *e*) 80 *f*) 80 *g*) 90 *h*) 800
6. *a*) 12 *b*) 9 *c*) 6 *d*) 4 *e*) 5
7. *a*) 1.78 *b*) 2.51 *c*) 2.24 *d*) 2.51 *e*) 3.16 *f*) 3.84

Answers

36. Roots rules (Page 40)

1. *a*) 2 *b*) 4 *c*) 2 *d*) 2 *e*) 2 *f*) 5 *g*) 10 *h*) 2
2. *a*) $2\sqrt{2}$ *b*) $3\sqrt{2}$ *c*) $2\sqrt[5]{4}$ *d*) $3\sqrt[3]{2}$ *e*) $2\sqrt[3]{4}$
   f) $10\sqrt[4]{100}$ *g*) $10\sqrt{2}$ *h*) $10\sqrt[3]{10}$ *i*) $10\sqrt[5]{10}$
   j) $8\sqrt{2}$ *k*) $2\sqrt[3]{4}$ *l*) $5\sqrt[3]{4}$
3. *a*) 5 *b*) 2 *c*) $5\sqrt{5}$ *d*) 5 *e*) 10 *f*) $\frac{1}{5}$
   g) $\frac{1}{2}$ *h*) 2 *i*) 2 *j*) $\frac{1}{2}$ *k*) $\frac{1}{2\sqrt{2}}$ *l*) $\frac{1}{5}$
4. *a*) $\frac{3}{5}$ *b*) $\frac{2}{3}$ *c*) $\frac{2}{3}$ *d*) $\frac{5}{4}$ *e*) $\frac{10}{3}$ *f*) $\frac{5}{3}$
5. *a*) $2\sqrt{3}$ *b*) $\sqrt[3]{25}$ *c*) $9\sqrt{3}$ *d*) $5\sqrt[3]{5}$ *e*) $\sqrt[4]{100}$ *f*) $\sqrt[3]{25}$
6. *a*) 2 *b*) 5 *c*) 20 *d*) 10 *e*) 10 *f*) 5
7. *a*) 3 *b*) $\frac{1}{10}$ *c*) 2 *d*) 2 *e*) $\frac{1}{2}$ *f*) $\frac{3}{2}$ *g*) $\frac{2}{5}$ *h*) $\frac{6}{5}$
   i) $\frac{3}{10}$ *j*) $\frac{3}{2}$ *k*) $\frac{1}{2}$ *l*) $\frac{1}{6}$ *m*) $\frac{1}{10}$ *n*) $\frac{1}{20}$ *o*) $\frac{1}{20}$ *p*) $\frac{3}{20}$

37. Equations with powers and radicals (Page 41)

1. *a*) *yes* *b*) *yes* *c*) *no* *d*) *no* *e*) *yes* *f*) *yes*
   g) *yes* *h*) *yes* *i*) *no*
2. *a*) 5 *b*) *no solutions* *c*) 5 *d*) 8 *e*) 10
   f) 2 *g*) 3 *h*) 2 *i*) 5 *j*) 2
3. *a*) 4 *b*) 2 *c*) 5 *d*) 2 *e*) 10 *f*) 2 *g*) 4 *h*) 2
   i) 6 *j*) 10
4. *a*) 4 *b*) 8 *c*) 9 *d*) 16 *e*) 1 *f*) 27 *g*) 8
   h) 1 *i*) 125 *j*) 10000
5. *a*) 2 *b*) 3 *c*) *any odd natural number* *d*) 6
   e) 3 *f*) 10 *g*) 3 *h*) 3
6. *a*) 2 *b*) 3 *c*) 4 *d*) 5 *e*) 6
7. *a*) 5 *b*) 2 *c*) 5 *d*) 3 *e*) 3 *f*) 512 *and* 8
   g) 3 *h*) 2 *i*) 64 *j*) 27 *k*) 4 *l*) 10

38. The link between radicals and powers (Page 42)

1. *a*) $2^{\frac{1}{2}}$ *b*) $27^{\frac{1}{3}}$ *c*) $5^{\frac{1}{4}}$ *d*) $2^{\frac{1}{5}}$ *e*) $64^{\frac{1}{4}}$ *f*) $1024^{\frac{1}{10}}$
   g) *not defined*
2. *a*) $\sqrt{2}$ *b*) $\sqrt{3}$ *c*) $\sqrt[3]{8}$ *d*) $\sqrt{10}$ *e*) $\sqrt[3]{27}$ *f*) $\sqrt{100}$
   g) $\sqrt[4]{625}$
3. *a*) $2^{\frac{3}{2}}$ *b*) $7^{\frac{2}{3}}$ *c*) $5^{\frac{2}{5}}$ *d*) $10^{\frac{5}{6}}$ *e*) $7^{\frac{3}{7}}$ *f*) 3^2 *g*) $2^{\frac{2}{3}}$
4. *a*) $\sqrt[5]{3^3}$ *b*) $5\ \sqrt{5}$ *c*) $10\ \sqrt[4]{10}$ *d*) $\sqrt[3]{9^2}$ *e*) $7\ \sqrt[3]{7}$
   f) $\sqrt[5]{20^4}$ *g*) $5\ \sqrt[3]{25}$
5. *a*) 9 *b*) 8 *c*) 5 *d*) 8 *e*) 4 *f*) $\sqrt{2}$ *g*) $3\sqrt{3}$
6. *a*) $\sqrt[6]{3^5}$ *b*) $\sqrt[12]{10^7}$ *c*) $\sqrt[15]{5^8}$ *d*) $\sqrt[4]{2^3}$ *e*) $\sqrt[20]{2^9}$
   f) $10\ \sqrt[12]{10}$ *g*) $\sqrt[12]{2^{11}}$ *h*) $3\ \sqrt[60]{3^{17}}$
7. *a*) $\sqrt[6]{2}$ *b*) $\sqrt[12]{10}$ *c*) $\frac{1}{\sqrt[6]{5}}$ *d*) $\frac{1}{\sqrt[20]{2}}$ *e*) $\sqrt[12]{5}$ *f*) $\frac{1}{\sqrt[6]{10}}$
   g) $\sqrt[4]{3}$
8. *a*) $\sqrt[6]{2}$ *b*) $\sqrt[6]{2}$ *c*) $\sqrt[12]{10}$ *d*) $\sqrt[4]{5}$ *e*) $\sqrt[9]{10}$ *f*) $\sqrt[15]{4}$
   g) $\sqrt[12]{64}$
9. *a*) $\sqrt[12]{2^5}$ *b*) $\sqrt[12]{10^7}$ *c*) $\sqrt[60]{3^7}$ *d*) $\frac{1}{\sqrt[60]{2^{13}}}$ *e*) $\frac{1}{\sqrt[6]{3}}$ *f*) $\sqrt[8]{2}$
   g) $\sqrt[12]{10}$ *h*) $\frac{1}{\sqrt[3]{5}}$ *i*) $\sqrt[3]{5^2}$ *j*) $\sqrt[24]{5}$

39. Order of operations (V) (Page 43)

1. *a*) 3 *b*) 2 *c*) 4 *d*) 2 *e*) 4 *f*) 3
2. *a*) 1 *b*) 2 *c*) 2 *d*) 2 *e*) 8 *f*) 0
3. *a*) 15 *b*) 2 *c*) 1 *d*) $\frac{4}{5}$ *e*) 2 *f*) 3
   g) 1000 *h*) 2 *i*) 1000 *j*) $4\ \sqrt{2}$ *k*) $\sqrt{2}$
   l) 4 $(\sqrt{2}\ \ \sqrt{3})$ *m*) 3 *n*) $\frac{3}{4}$
4. *a*) 0.503 *b*) 2.310 *c*) 2.070 *d*) 3.148

Answers

40. Substitution (Page 44)

1. $a)\ no \quad b)\ yes \quad c)\ yes \quad d)\ yes \quad e)\ no$
   $f)\ no \quad g)\ yes \quad h)\ no \quad i)\ yes \quad j)\ no$

2. $a)\quad 8 \quad b)\ \frac{1}{4} \quad c)\quad 4 \quad d)\ 2 \quad e)\quad 4$
   $f)\quad 1 \quad g)\ 1 \quad h)\ 2 \quad i)\ \frac{1}{2} \quad j)\ 0$

3. $a)\quad 3 \quad b)\quad 8 \quad c)\quad 2 \quad d)\ \frac{1}{2} \quad e)\ 1$
   $f)\ 4 \quad g)\ \frac{1}{9} \quad h)\ \frac{2}{3} \quad i)\ 36 \quad j)\quad 1$

4. $a)\ z \quad 11, u \quad 8, v \quad 2 \quad b)\ d \quad 25, e \quad 3, f \quad 225$
   $c)\ A \quad 62, V \quad 30, D \quad \sqrt{38}$

5. $a)\quad 3 \quad b)\ 8 \quad c)\ 6 \quad d)\ 2 \quad e)\ 1 \quad f)\ 0 \quad g)\ 2 \quad h)\quad 54$

6. $a)\ 0 \quad b)\ 1 \quad c)\ 1 \quad d)\quad 3 \quad e)\ 0 \quad f)\ 0 \quad g)\ 16 \quad h)\ 1$