9.5 Composition of Functions

A Composition of Functions

Any function $y = f(x)$ can be imagined as a machine f who changes the input x into the output y:

$$x \to \boxed{\ f\ } \to y$$

Let connect now the output of the function machine g as the input of the function machine f corresponding to a different function $z = f(y)$:

$$x \to \boxed{\ g\ } \to y \to \boxed{\ f\ } \to z$$

Then, the following expressions can be written:

$$y = g(x),\ z = f(y),\ z = f(g(x))$$

So, we can replace the machines g and f by a new machine $f \circ g$ who changes the input x directly into the output z:

$$x \to \boxed{\ f \circ g\ } \to z$$

So, the relation

$$(f \circ g)(x) = f(g(x))$$

defines the composition of the functions f and g. Similarly:

$$(g \circ f)(x) = g(f(x))$$
$$(f \circ f)(x) = f(f(x))$$
$$(g \circ g)(x) = g(g(x))$$

Ex 1. Consider the following functions $f(x) = x + 1$, $g(x) = x^2 - 2$, $h(x) = \sqrt{x} + 3$. Find:

a) $(f \circ g)(0) = f(g(0)) = f(-2)$
$= -1$

b) $(g \circ f)(2) = g(f(2)) = g(3)$
$= 7$

c) $(f \circ h)(-1) = f(h(-1)) = f(\sqrt{-1} + 3)$
$=$ undefined

d) $(h \circ f)(0) = h(f(0)) = h(1)$
$= 4$

e) $(f \circ f)(2) = f(f(2)) = f(3)$
$= 4$

f) $(g \circ g)(1) = g(g(1)) = g(-1)$
$= -1$

g) $(h \circ h)(1) = h(h(1)) = h(4)$
$= 5$

i) $(f \circ g \circ h)(1) = f(g(h(1)))$
$= f(g(4)) = f(14)$
$= 15$

B Domain and Range

The domain of $f \circ g$ is a subset of the domain of g:

$$D_{f \circ g} = \{x \in D_g \mid g(x) \in D_f\}$$

The domain of $f \circ g$ consists of the numbers x in the domain of g such that $g(x)$ is in the domain of f.
The range of $f \circ g$ is a subset of the range of f.

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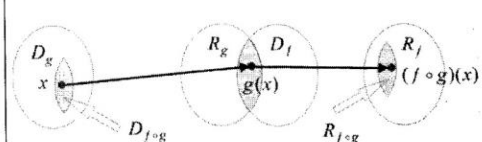

Ex 2. The functions f and g are given by the following mapping diagrams. Find each composition and then find the domain and the range of the composition.

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f: $1 \rightarrow 2$, $2 \rightarrow -1$, $3 \rightarrow 0$

g: $-1 \rightarrow -3$, $0 \rightarrow 2$, $1 \rightarrow 3$

a) $f \circ g$

b) $g \circ f$

c) $f \circ f$

d) $g \circ g$

Ex 3. Consider the following functions $f(x) = 2x - 1$, $g(x) = x^2$, $h(x) = \sqrt{x-1}$, and $k(x) = \frac{1}{x+1}$. Find the following compositions ~~and for each case state the domain and the range.~~

a) $(f \circ g)(x) = f(x^2)$

$= 2(x^2) - 1$

$= 2x^2 - 1$

b) $(k \circ k)(x) = k(k(x))$

$= k\left(\frac{1}{x+1}\right)$

$= \dfrac{1}{\frac{1}{x+1} + 1}$

$= \dfrac{x+1}{x+2}$

c) $(f \circ g \circ h)(x) = f(g(h(x)))$

$= f(g(\sqrt{x-1}))$

$= f(x+1) = 2(x+1) - 1$

$= 2x + 1$

d) $(k \circ g \circ f)(x) = k(g(f(x)))$

$= k(g(2x-1))$

$= k((2x-1)^2) = \dfrac{1}{(2x-1)^2 + 1}$

Ex 4. Prove that:

a) $(f \circ f^{-1})(x) = x$

$y \underset{f^{-1}}{\overset{f}{\rightleftarrows}} x$

b) $(f^{-1} \circ f)(x) = x$

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(y) = x$

Ex 5. Prove that:

$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

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Reading: Nelson Textbook, Pages 545-551
Homework: Nelson Textbook, Page 552: #5aef, 6ace, 7ac, 9, 13, 16