### A Definitions

The sum and difference of two functions are defined by

\[
(f + g)(x) = f(x) + g(x) \\
(f - g)(x) = f(x) - g(x)
\]

#### Ex 1. Let \(f(x) = x^2 - 1\) and \(g(x) = 1 + \sqrt{x+1}\). Find

a) \((f + g)(0) = \frac{f(0)}{0} + g(0)\)
   \[
   = -1 + 2 = 1
   \]

b) \((f - g)(3) = f(3) - g(3)\)
   \[
   = 8 - 3 = 5
   \]

c) \((f + g)(-2) = f(-2) + g(-2)\)
   \[
   = 3 + \text{undefined} = \text{undefined}
   \]

d) \((f + g)(x) = x^2 - 1 + 1 + \sqrt{x+1}\)
   \[
   = x^2 + \sqrt{x+1}
   \]

### B Domain of Sum and Difference of Two Functions

The domain of the sum of two functions is the intersection of their domains.

\[
D_{f \pm g} = D_f \cap D_g
\]

#### Ex 2. For each case, find the domain of the sum of the given functions.

a) \(f(x) = x\); \(g(x) = 1/x^2\)
   \[
   \mathbb{R}, \quad x \neq 0
   \]
   \[
   D_{f \pm g} = \mathbb{R} - \{0, y\}
   \]

b) \(f(x) = \sqrt{2-x}\); \(g(x) = \log(x+1)\)
   \[
   2 - x \geq 0 \quad x + 1 > 0
   \]
   \[
   x \leq 2 \quad x > -1
   \]
   \[
   D_{f \pm g} = (-1, 2]
   \]

c) \(f(x) = 2^{-x^2}\); \(g(x) = x^3 + 1\)
   \[
   \mathbb{R}
   \]
   \[
   D_{f \pm g} = \mathbb{R}
   \]

d) \(f(x) = \sqrt{4-x^2}\); \(g(x) = \frac{1}{x-1}\); \(h(x) = \sqrt{x}\)
   \[
   4 - x^2 \geq 0 \quad x \neq 1 \quad x > 0
   \]
   \[
   x^2 \leq 4 \quad -2 \leq x \leq 2
   \]
   \[
   D_{f \pm g + h} = (0, 1) \cup (1, 2]
   \]
C Point by Point
Evaluate \( f \pm g \) at every possible number \( x \).

Ex 3. Given
\[
\begin{array}{c|ccccc}
X & -1 & 0 & 1 & 2 \\
\hline
f(x) & 2 & -1 & 0 & -1 \\
g(x) & -1 & 3 & -1 & -5 \\
\hline
(f \pm g)(x) & 1 & 2 & -1 & -6
\end{array}
\]
Find \( f + g \).
\[
f + g = \begin{cases} (0,0), & (-2,3) \end{cases}
\]

Ex 4. The functions \( f \) and \( g \) are given by their graphs on the right figure. Graph the function \( f - g \).

\[
y = (f - g)(x)
\]

Ex 5. Complete the following table. Justify your reasoning.

<table>
<thead>
<tr>
<th></th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f + g )</td>
<td>even</td>
</tr>
<tr>
<td>even</td>
<td>even</td>
</tr>
<tr>
<td>odd</td>
<td>neither</td>
</tr>
<tr>
<td>neither</td>
<td>neither</td>
</tr>
</tbody>
</table>

Ex 6. For each case, justify your answer.

a) Is the sum of two polynomial functions a polynomial function?
   Yes

b) Is the difference of two rational functions a rational function?
   Yes

c) Is the sum of two sine functions a sine function?
   Yes only if their periods are equal.
   No if their periods are not equal.
**Ex 7.** Write the following functions as a sum or a difference of two other functions.

| a) \( f(x) = \frac{1}{x^2 - 9} \) & \( b) \ f(x) = \log \frac{x}{x+1} = \log x - \log (x+1) \) |
|---|---|
| \( f(x) = \frac{A}{x-3} + \frac{B}{x+3} \) & \( c) \ f(x) = \sin(x - \pi/4) \) |
| \( 0 = A + B \Rightarrow B = -A \) & \( \frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \cos x \) |
| \( 1 = 3A - 3B \Rightarrow 1 = 6A \) & |
| \( f(x) = \frac{1}{6(x-3)} - \frac{1}{6(x+3)} \) & |

**Ex 8.** Is the sum of two increasing functions, increasing, decreasing or neither? Give examples to justify your answer.

- **Increasing**
  - \( f(x) = x \) (Increasing)
  - \( g(x) = x^3 \) (Increasing)
  - \((f+g)(x) = x + x^3\) is increasing.

**Ex 9.** If \( Z_f = \{1, 2, 3\} \) is the set of all zeros of the function \( f \) and \( Z_g = \{0, 1, 2\} \) is the set of all zeros of the function \( g \), what could you say about the set of all zeros of the function \( f + g \)? Explain your reasoning.

- \( Z_f \cap Z_g \subseteq Z_{f+g} \)
- \( \sqrt{1, 2} \subseteq Z_{f+g} \)
- At least 1 and 2 are zeros of \( f + g \).
- More are possible.

**Ex 10.** Let \( f(x) = \sin x \) and \( g(x) = \cos x \). Write \( f + g \) and \( f - g \) as:

- a) a single sine function
  - \( (f+g)(x) = \sin x + \cos x = \sqrt{2} \sin (x + \pi/4) \)
  - \( (f-g)(x) = \sin x - \cos x = \sqrt{2} (\sin x - \cos x) \)

- b) a single cosine function
  - Subtract \( \frac{\pi}{2} \) from previous
  - \( \sin x + \cos x = \sqrt{2} \cos \left( x - \frac{\pi}{4} \right) \)
  - \( \sin x - \cos x = \sqrt{2} \sin \left( x - \frac{3\pi}{4} \right) \)

**Ex 11.** The rational function \( y = f(x) \) has a horizontal asymptote \( y = 5 \) and the rational function \( y = g(x) \) has a horizontal asymptote \( y = -3 \). What could you say about the horizontal asymptote of the functions:

- a) \( f + g \)
  - \( \text{HA: } y = 5 + (-3) = 2 \)

- b) \( f - g \)
  - \( \text{HA: } y = 5 - (-3) = 8 \)

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**Reading:** Nelson Textbook, Pages 521-528

**Homework:** Nelson Textbook, Page 528 #3, 4, 5, 7, 10, 13, 16

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9.2 Combining Two Functions: Sums and Differences
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