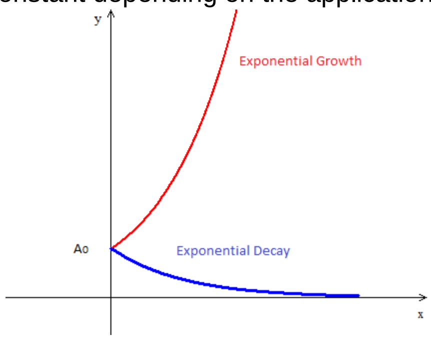
8.7 Solving Problems with Exponential and Logarithmic Functions

A Exponential Growth and Decay

Exponential Growth and Decay may be modelled by a function of the form:

$$A(t) = A_0(b^{kt})$$

where
t is time
A_0 is the initial amount
$A(t)$ is the amount at time t
b is the base
k is a constant depending on the application



Ex 1. Let $f(x) = 4(2^{4x-1})$. **Skip This**

a) Write this relation in the form $f(x) = Ab^{Bx}$.

b) Write this relation in the form $f(x) = Ab^{x}$.

c) Write this relation in the form $f(x) = A(10^{Bx})$.

d) Write this relation in the form $f(x) = A(3^{Bx})$.

B Common Ratio **Skip This**

The values of the exponential growth function form a geometric sequence:

$$\frac{y_2}{y_1} = \frac{y_3}{y_2} = \ldots = \frac{y_{n+1}}{y_n}$$

where

$y_1 = f(x_1)$, $y_2 = f(x_2)$, …

and x_1, x_2, … are in arithmetic sequence.

Ex 2. Show that the following relation is exponential.

Skip This

x	y
1	2
2	6
3	18
4	54
5	162

C Developing Exponential Growth Formula

If r is the increasing rate per year, use:

$$A(t) = A_0(1+r)^t$$

Indeed $A(1) = A_0(1+r)^1 = A_0 + rA_0$.

If, over a period T, the amount is increasing b times, use:

$$A(t) = A_0 b^{\frac{t}{T}}$$

Indeed $A(T) = A_0 b^{\frac{T}{T}} = bA_0$.

Ex 3. For each case, find an exponential function that model best the situation.

a) The value of a house is increasing by 7% per year.

$V(t) = V_0(1+7/100)^t$

b) The number of bacteria is triple every two hours.

$N(t) = N_0\, 3^{\frac{t}{2}}$

c) The number of bacteria is double every five hours.

$N(t) = N_0\, 2^{\frac{t}{5}}$

D Developing Exponential Decay Formula

Exponential Decay may be modelled by a function of the form:

$$A(t) = A_0(b^{kt})$$

or by

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{H}}$$

where H is half-life

$$A(H) = \frac{A_0}{2}$$

or by

$$A(t) = A_0(1-r)^t$$

where r is the decreasing rate per year

$$A(1) = A_0(1-r)^1 = A_0 - rA_0$$

Ex 4. For each case, find an exponential function that model best the situation.

a) The value of a car is decreasing by 5% per year.

$$V(t) = V_0(1-5/100)^t$$

b) The half-life of a radioactive source is 81 years.

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{81}}$$

c) The luminosity decreases 3 times for each 10cm of depth.

$$L(h) = L_0\left(\frac{1}{3}\right)^{\frac{d}{10}}$$

E Sound Level

$$L = 10\log\left(\frac{I}{I_0}\right)$$

where

L is the soundness (sound level) in decibels

I is the intensity of the sound

$I_0 = 10^{-12} W/m^2$ is a constant (intensity of the sound at the threshold of hearing)

Note. $L_2 - L_1 = 10\log(I_2/I_1)$

Ex 5. A whisper has a sound level of 15 dB and a rock concert has a sound level of 120 dB. How many more intense is the rock concert in comparison to a whisper?

See the solution below

F Earthquake Magnitude

$$M = \log\left(\frac{A}{A_0}\right)$$

where

M is the magnitude of the earthquake

A is the amplitude (intensity) of the earthquake

A_0 is a constant

Note. $M_2 - M_1 = \log(A_2/A_1)$

Ex 6. In 2017, in Mexico, two earthquakes happened with a magnitude more than 7. One happened on September 7 and had a magnitude of 8.2 and the other happened on September 19 and had a magnitude of 7.1. How many times was the amplitude of the September 7 earthquake greater in comparison to the amplitude of the September 19 earthquake?

See the solution below

G pH Scale

$$pH = -\log n$$

where

pH is a number measuring acidity/alkalinity of a substance

$n = [H^+]$ is the concentration of hydrogen ions

Ex 7. Lemon juice has a pH of 2.5 and milk has a pH of 9. How many times the hydrogen ions are more concentrated in lemon juice than in milk.

See the solution below

Reading: Nelson Textbook, Pages 493-499
Homework: Nelson Textbook, Page 499 #1-5, 8, 10, 14, 15, 17, 18

Ex 5

Whisper

$$15 = 10 \log_{10} \frac{I_W}{I_0}$$

$$10^{15/10} = \frac{I_W}{I_0}$$

Rock Concert

$$120 = 10 \log_{10} \frac{I_{RC}}{I_0}$$

$$10^{120/10} = \frac{I_{RC}}{I_0}$$

$$\frac{10^{120/10}}{10^{15/10}} = \frac{I_{RC}/I_0}{I_W/I_0} = \frac{I_R}{I_W}$$

$$\frac{I_R}{I_W} = \frac{10^{12}}{10^{1.5}} = 10^{12-1.5} = 10^{10.5}$$

$$\simeq 31,622,776,600$$

$\therefore$ The intensity of the sound at a rock concert is about 31 billions times greater in comparisson to the whisper

Ex 6

Sep 7 | Sep 19

$8.2 = \log \frac{A_1}{A_0}$ $\quad 7.1 = \log \frac{A_2}{A_0}$

$10^{8.2} = \frac{A_1}{A_0}$; $10^{7.1} = \frac{A_2}{A_0} \Rightarrow \frac{10^{8.2}}{10^{7.1}} = \frac{A_1}{A_2}$

$\frac{A_1}{A_2} = 10^{8.2-7.1} = 10^{1.1} \simeq 12.6$

∴ The amplitude (intensity) of Sep 7 earthquake was 12.6 times greater than Sep 19 earthquake.

Ex 7

Lemon Juice | Milk

$2.5 = -\log n_1$ $\quad 9 = -\log n_2$

$10^{-2.5} = n_1$; $10^{-9} = n_2 \Rightarrow$

$\frac{n_1}{n_2} = \frac{10^{-2.5}}{10^{-9}} = 10^{9-2.5} = 10^{6.5}$

$\simeq 3,162,277$

∴ in lemon juice there are about 3 millions times more hydrogen ions than in milk.