8.5 Solving Exponential Equations

A One-to-one property The exponential function is *one-to-one* function. So: $a^x = a^y \quad \Leftrightarrow \quad x = y$ $a > 0, a \neq 1, x \in R, y \in R,$	Ex 1. Solve the following exponential equations by using the one-to-one property. a) $2^x = 64 \quad \Rightarrow \quad 2^x = 2^6 \quad \Rightarrow \quad x = 6$ b) $10^{2x-3} = 0.0001 \Rightarrow 10^{2x-3} = 10^{-4} \Rightarrow 2x - 3 = -4$ $x = -1/2$ c) $2^{-x} = \sqrt[5]{16} \quad \Rightarrow \quad 2^{-x} = 2^{4/5} \quad \Rightarrow$ $x = -4/5$ d) $8^x = \sqrt[3]{0.0625} \quad \Rightarrow \quad 2^{3x} = (1/16)^{1/3}$ $\Rightarrow 2^{3x} = 2^{-4/3} \quad \Rightarrow \quad x = -4/9$
B Change of Variable Sometimes, *changing of the variable* may help solving the exponential equation. For example: $a^x = y; \quad y > 0$	**Skip This Topic**
Ex 2. Use the change of variable method to solve each of the following exponential equations. a) $2^x + 2^{-x} = 4.25$ **Skip This** b) $5 \cdot 2^x - 4^x + 24 = 0$ **Skip This**	c) $\dfrac{2^x - 2^{-x}}{2^x + 2^{-x}} = -\dfrac{63}{65}$ **Skip This** d) $2^{x+1} + 2^{2x} = 2^x + 2 + \sqrt{2}$ **Skip This**

C Logarithms Sometimes, *logarithms* are needed in order to solve exponential equations.	Ex 3. Solve each equation using logarithms. a) $2^{3x-1}=5$ **See the solution below** b) $3^{x-1}=4^{x+1}$ **See the solution below**
D Applications Many *applications* are related to solving exponential equations.	
Ex 4. A species of bacteria doubles each 10 minutes. The initial number of bacteria is 200. a) Find the exponential function describing the bacteria population growth. **See the solution below** b) Find the bacteria population after one hour. c) Find the time (in minutes) after which the bacteria population is 123456.	Ex 5. A 100 g sample of plutonium-238 has a half-life of 88 years. a) Find the exponential function describing the radioactive decay. **See the solution below** b) Find the mass of radioactive source after 10 years. c) Find the time (in years) after which the mass of the radioactive source will be $3.21\,g$.

Reading: Nelson Textbook, Pages 480-484
Homework: Nelson Textbook, Page 485: #5, 7, 8, 10, 12, 14, 15, 16, 17

Ex 3a Solve: $2^{3x-1} = 5$

Method #1

a) Change the exponential form into a logarithmic form:

$$\log_2 5 = 3x - 1$$

b) Solve for x:

$$x = \underbrace{\frac{1 + \log_2 5}{3}}_{\text{exact value}} \simeq \underbrace{1.107}_{\text{approximate value}}$$

Method #2

a) Take log of both sides

$$\log 2^{3x-1} = \log 5$$

b) Use the power law

$$(3x-1)\log 2 = \log 5$$

c) Solve for x

$$x = \frac{1 + \frac{\log 5}{\log 2}}{3} \simeq 1.107$$

Ex 3b Solve: $3^{x-1} = 4^{x+1}$

Method #1

a) Rewrite the equation

$$\frac{3^x}{3} = 4 \cdot 4^x \Rightarrow \left(\frac{3}{4}\right)^x = 12$$

b) change to logarithmic form

$$\underbrace{\log_{3/4} 12 = x}_{\text{exact value}} \simeq \underbrace{-8.638}_{\text{approximate value}}$$

Method #2

a) Apply log to both sides

$$\log 3^{x-1} = \log 4^{x+1}$$

b) Use the power law

$$(x-1)\log 3 = (x+1)\log 4$$

c) isolate the variable x

$$x(\log 3 - \log 4) = \log 4 + \log 3$$

d) solve for x

$$\underbrace{x = \frac{\log 4 + \log 3}{\log 3 - \log 4}}_{\text{exact value}} \simeq -8.638$$

Ex 4

a) $$P(t) = \underbrace{200}_{\text{initial value}} \cdot 2^{\frac{t}{10}}$$

time (in minutes)

doubling period

b) $t = 1 \text{ hour} = 60$ minutes

$$P(60) = 200 \cdot 2^{\frac{60}{10}} = 12{,}800$$

∴ After one hour, the bacteria population is 12,800

c) Find t when $P(t) = 123456$

$$123456 = 200 \cdot 2^{t/10}$$

$$\frac{123456}{200} = 2^{t/10}$$

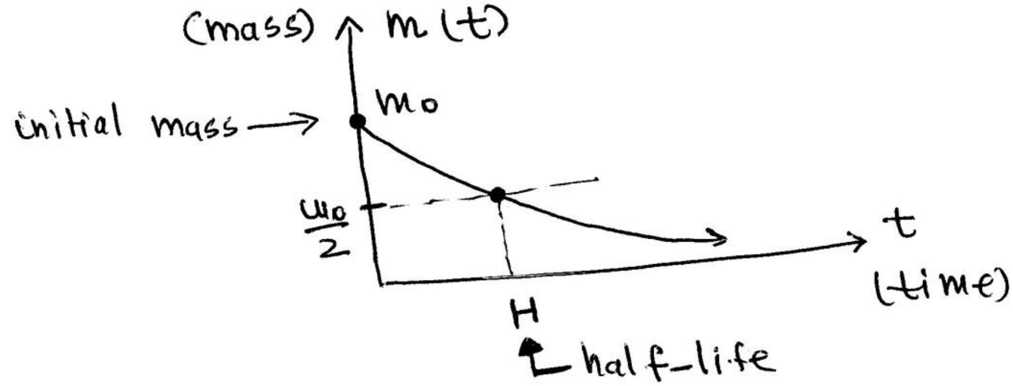
$$\frac{t}{10} = \log_2 \frac{123456}{200} = \log_2 617.28$$

$$t = 10 \cdot \log_2 617.28$$

$$= 10 \cdot \frac{\log 617.28}{\log 2} \simeq 92.698$$

∴ After 92.698 minutes, the bacteria population is 123456

Ex 5



a) $m(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{H}}$

$m(t) = (100)\,0.5^{t/88}$

(m in grams) (t in years)

b) $t = 10$ years

$m(10) = (100)\,0.5^{\frac{10}{88}} \simeq 92.43$

$\therefore$ After 10 years, only 92.43 g of the sample is radioactive

c) Find t when $m(t) = 3.21\,g$

$3.21 = (100)\,0.5^{t/88}$

$\frac{3.21}{100} = 0.5^{t/88} \Rightarrow \log\frac{3.21}{100} = \frac{t}{88}\log 0.5$

$t = 88\,\frac{\log 0.0321}{\log 0.5} \simeq 436.59$

$\therefore$ It takes 436.59 years for the sample to have only 3.21 g radioactive