# 8.5 Solving Exponential Equations

## A One-to-one property

The exponential function is *one-to-one* function. So:

\[
\begin{align*}
a^x &= a^y \iff x = y \\
a > 0, a \neq 1, x \in R, y \in R,
\end{align*}
\]

<table>
<thead>
<tr>
<th>Ex 1. Solve the following exponential equations by using the one-to-one property.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) (2^x = 64 \Rightarrow 2^x = 2^6 \Rightarrow x = 6)</td>
</tr>
<tr>
<td>b) (10^{2x-3} = 0.0001 \Rightarrow 10^{2x-3} = 10^{-4} \Rightarrow 2x - 3 = -4) (x = 1/2)</td>
</tr>
<tr>
<td>c) (2^{-x} = \sqrt[4]{16} \Rightarrow 2^{-x} = 2^{4/5} \Rightarrow x = -4/5)</td>
</tr>
<tr>
<td>d) (8^x = 3^{0.0625} \Rightarrow 2^{3x} = (1/16)^{1/3} \Rightarrow 2^{3x} = 2^{-4/3} \Rightarrow x = -4/9)</td>
</tr>
</tbody>
</table>

## B Change of Variable

Sometimes, *changing of the variable* may help solving the exponential equation. For example:

\[
a^x = y, \quad y > 0
\]

<table>
<thead>
<tr>
<th>Skip This Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ex 2. Use the change of variable method to solve each of the following exponential equations.</td>
</tr>
<tr>
<td>a) (2^x + 2^{-x} = 4.25)</td>
</tr>
<tr>
<td>b) (5 \cdot 2^x - 4^x + 24 = 0)</td>
</tr>
<tr>
<td>c) (\frac{2^x - 2^{-x}}{2^x + 2^{-x}} = \frac{63}{65})</td>
</tr>
<tr>
<td>d) (2^{x+1} + 2^{2x} = 2^x + 2 + \sqrt{2})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Skip This</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skip This</td>
</tr>
<tr>
<td>Skip This</td>
</tr>
<tr>
<td>Skip This</td>
</tr>
</tbody>
</table>
### C Logarithms

Sometimes, *logarithms* are needed in order to solve exponential equations.

Ex 3. Solve each equation using logarithms.

- a) \(2^{3x-1} = 5\)
  
  **See the solution below**

- b) \(3^{x-1} = 4^{x+1}\)
  
  **See the solution below**

### D Applications

Many *applications* are related to solving exponential equations.

Ex 4. A species of bacteria doubles each 10 minutes. The initial number of bacteria is 200.

- a) Find the exponential function describing the bacteria population growth.
  
  **See the solution below**

- b) Find the bacteria population after one hour.

- c) Find the time (in minutes) after which the bacteria population is 123456.

Ex 5. A 100 g sample of plutonium-238 has a half-life of 88 years.

- a) Find the exponential function describing the radioactive decay.
  
  **See the solution below**

- b) Find the mass of radioactive source after 10 years.

- c) Find the time (in years) after which the mass of the radioactive source will be 3.21 g.

**Reading:** Nelson Textbook, Pages 480-484

**Homework:** Nelson Textbook, Page 485: #5, 7, 8, 10, 12, 14, 15, 16, 17
Ex 3a  Solve: \[ 2^{3x-1} = 5 \]

**Method #1**

a) Change the exponential form into a logarithmic form:

\[ \log_2 5 = 3x - 1 \]

b) Solve for \( x \):

\[ x = \frac{1 + \log_2 5}{3} \approx 1.107 \text{ approximate value} \]

\[ \text{exact value} \]

**Method #2**

a) Take \( \log \) of both sides

\[ \log 2^{3x-1} = \log 5 \]

b) Use the power law

\[ (3x-1) \log 2 = \log 5 \]

c) Solve for \( x \)

\[ x = \frac{1 + \frac{\log 5}{\log 2}}{3} \approx 1.107 \]
Ex 3b

Solve: $3^{x-1} = 4^{x+1}$

**Method #1**

a) Rewrite the equation

$$\frac{3^x}{3} = 4 \cdot 4^x \Rightarrow \left(\frac{3}{4}\right)^x = 12$$

b) Change to logarithmic form

$$\log_{3/4} 12 = x \approx -8.638$$

**Method #2**

a) Apply $\log_4$ to both sides

$$\log_4 3^{x-1} = \log_4 4^{x+1}$$

b) Use the power law

$$(x-1)\log_4 3 = (x+1)\log_4 4$$

c) Isolate the variable $x$

$$x \left(\log_4 3 - \log_4 4\right) = \log_4 4 + \log_4 3$$

d) Solve for $x$

$$x = \frac{\log_4 4 + \log_4 3}{\log_4 3 - \log_4 4} \approx -8.638$$

**Exact value**
Ex 4

a) \[ P(t) = 200 \cdot 2^{\frac{t}{10}} \]

\[ \text{initial value} \]

b) \[ t = 1 \text{ hour} = 60 \text{ minutes} \]
\[ P(60) = 200 \cdot 2^{\frac{60}{10}} = 12,380 \]

\[ \therefore \text{After one hour, the bacteria population is 12,380} \]

c) Find \( t \) when \[ P(t) = 12,345,656 \]
\[ 12,345,656 = 200 \cdot 2^{\frac{t}{10}} \]
\[ \frac{12,345,656}{200} = 2^{\frac{t}{10}} \]
\[ \frac{t}{10} = \log_2 \frac{12,345,656}{200} = \log_2 617.28 \]
\[ t = 10 \cdot \log_2 617.28 \]
\[ = 10 \cdot \frac{\log 617.28}{\log 2} \approx 92.698 \]

\[ \therefore \text{After 92.698 minutes, the bacteria population is 12,345,656} \]
Ex 5

\[ m(t) = 100 \left( \frac{1}{2} \right)^{t/88} \]

\[ m(t) = \left(100\right) 0.5^{t/88} \]

(m in grams) (t in years)

b) \( t = 10 \) years

\[ m(10) = \left(100\right) 0.5^{\frac{10}{88}} \approx 92.43 \]

\text{After 10 years, only 92.43 g of the sample is radioactive.}

c) Find \( t \) when \( m(t) = 3.21 \) g

\[ 3.21 = \left(100\right) 0.5^{t/88} \]

\[ \frac{3.21}{100} = 0.5^{t/88} \]

\[ \log \frac{3.21}{100} = \frac{t}{88} \log 0.5 \]

\[ t = 88 \log \frac{0.0321}{0.5} \approx 436.59 \]

\text{It takes 436.59 years for the sample to have only 3.21 g radioactive.}