Exponential Function $\xleftrightarrow{\text{inverse function}}$ Logarithmic Function

8.1 Exploring the Logarithmic Function

A Logarithmic Functions

Logarithmic function is defined as the *inverse* function of the exponential function.

So, if $\boxed{y = f(x) = b^x}$ then $\boxed{x = f^{-1}(y)} = \log_b y$.

Note: b is the base of the exponential function and the base of the logarithmic function.

The following expressions are equivalent:

(exponential notation) $\boxed{y = b^x} \Leftrightarrow \boxed{x = \log_b y}$ (logarithmic notation)

Reading: "log(arithm) (to the) base b of y"

Note: $y = b^x$ is called the *exponential form* and $x = \log_b y$ is called the *logarithmic form*.

Ex 1. Convert the exponential form to the logarithmic form.

a) $8 = 2^3 \Longleftrightarrow 3 = \log_2 8$

b) $10000 = 10^4 \longrightarrow 4 = \log_{10} 10{,}000$

c) $0.00001 = 10^{-5} \longrightarrow -5 = \log_{10} 0.00001$

d) $1024 = 2^{10} \longrightarrow 10 = \log_2 1024$

Ex 2. Convert the logarithmic form to the exponential form.

a) $4 = \log_2 16 \Longleftrightarrow 2^4 = 16$

b) $3 = \log_{10} 1000 \longrightarrow 10^3 = 1000$

c) $-4 = \log_{10} 0.0001 \longrightarrow 10^{-4} = 0.0001$

d) $4 = \log_5 625 \longrightarrow 5^4 = 625$

B Domain, Range and other Restrictions

The domain and the range of the exponential function:

$$b^x : \underset{D}{(-\infty, +\infty)} \to \underset{R}{(0, +\infty)}$$

are interchanged to obtain the domain and the range of the logarithmic function:

$$\log_b x : \underset{D}{(0, +\infty)} \to \underset{R}{(-\infty, +\infty)}$$

(domain) $\boxed{D_{\log} = (0, \infty)}$

$\boxed{R_{\log} = \mathbb{R}}$ (range)

The base b satisfies the same restrictions from the exponential function:

$$\boxed{b > 0, b \neq 1}$$

Ex 3. Find if the following expressions are well defined.

a) $\log_{\sqrt{2}} 1$ (OK)

b) $\log_1 2$ (incorrect) $b = 1$

c) $\log_{\frac{1}{2}} \sqrt{2}$ (OK)

d) $\log_2(-10)$ (incorrect) $x = -10 < 0$

e) $\log_{-2} 3$ (incorrect) $b = -2$

C Basic Formulas

Ex 4. Use the exponential-logarithmic conversion to prove the following basic formulas:

a) $\boxed{\log_b 1 = 0} \Longleftrightarrow b^0 = 1$

b) $\boxed{\log_b b = 1} \Longleftrightarrow b^1 = b$

c) $\log_b \frac{1}{b} = -1 \Longleftrightarrow b^{-1} = \frac{1}{b}$

e) $\log_{\frac{1}{b}} b = -1 \qquad \left(\frac{1}{b}\right)^{-1} = b$

e) $\boxed{\log_b b^n = n} \qquad b^n = b^n$

D Basic Equations

Ex 5. Solve each equation by converting it to the exponential form.

a) $x = \log_5 25 = \log_5 5^2 = 2$

b) $x = \log_4 1 = 0$

c) $\log_x 16 = 2 \iff x^2 = 16$

$x = \pm 4$

$\therefore x = 4$

d) $\log_x 3 = \frac{1}{2} \iff x^{\frac{1}{2}} = 3 \Rightarrow x = 9$

$\therefore x = 9$

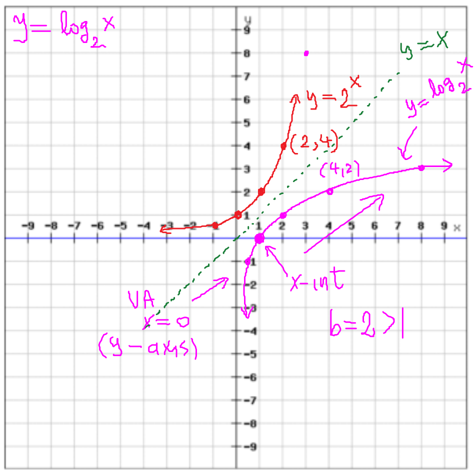
e) $\log_2 x = -2 \iff 2^{-2} = x$

$\therefore x = \frac{1}{4}$

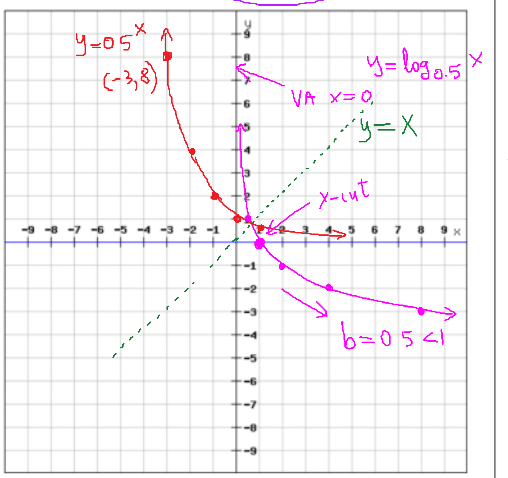
E Graph of the Logarithmic Function

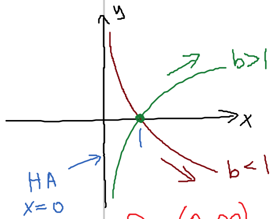
Ex 6. Graph on the grid provided below both $f(x) = 2^x$ and $f^{-1}(x) = \log_2 x$.

$y = 2^x$



Ex 7. Graph on the grid provided below both $f(x) = 0.5^x$ and $f^{-1}(x) = \log_{0.5} x = y \iff y = 0.5^x$





$D = (0, \infty)$

$R = \mathbb{R}$

F Characteristics of the Logarithmic Function

Ex 8. Use the graphs obtained at example 6 and 7 to conclude about the following characteristics of the logarithmic function.

$\boxed{y = \log_b x}$

- Domain: $(0, \infty)$
- Range: $\mathbb{R}$
- x-intercepts: $(1, 0)$
- y-intercepts: none

b > 1 b < 1

- Increasing/Decreasing:
- Horizontal Asymptotes: none
- Vertical Asymptotes: $x = 0$ (y-axis)
- Continuity: yes
- One-to-one: yes
- Key Points: $(1,0)$ $(b,1)$ $\left(\frac{1}{b}, -1\right)$

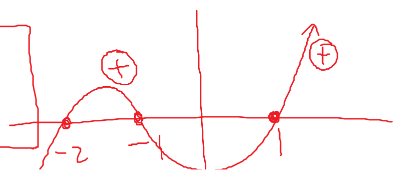
Reading: Nelson Textbook, Pages 448-450
Homework: Nelson Textbook, Page 451: #1ac 3, 4, 6, 8, 9

Ex Find the domain.

$$f(x) = \log_2 \underbrace{\frac{x^2-1}{x+2}}_{\text{input}}$$

$$\frac{x^2-1}{x+2} > 0 \longrightarrow (x-1)(x+1)(x+2) > 0$$

$\boxed{\therefore D = (-2, -1) \cup (1, \infty)}$



$\therefore D = (-2, -1) \cup (1, \infty)$