8.0 Exponential Function

A Exponents Laws

The following relations are called *exponent laws*:

$$(ab)^n = a^n b^n \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn} \qquad a^{-n} = \frac{1}{a^n}$$

Note. If the exponent is a *rational number* then:

$a^{\frac{p}{q}} = \sqrt[q]{a^p}$, q is a positive integer

B Definition

Exponential function $y = f(x)$ is defined by:

$$\boxed{y = f(x) = b^x}$$

Ex $f(x) = 2^x$

$f(x) = \pi^x$

where:

- b is the *base*
- x is the *argument* (input)
- y is the *value* of the exponential function (output)

Note. For each value of the base b a different exponential function is defined.

Ex 1. Find the *base* of the following exponential functions.

a) $y = 2^{3x} = (2^3)^x = 8^x \longrightarrow b = 8$

b) $y = \frac{1}{10^x} = (10^{-1})^x \longrightarrow b = 10^{-1} = \frac{1}{10} = 0.1$

c) $y = 3^{\frac{x}{2}} = (3^{\frac{1}{2}})^x = \sqrt{3}^x \rightarrow b = \sqrt{3}$

d) $y = 2^{-3x} = (2^{-3})^x = (\frac{1}{8})^x \rightarrow b = \frac{1}{8}$

e) $y = \sqrt{3^x} = (3^x)^{1/2} = (3^{1/2})^x \rightarrow b = \sqrt{3}$

C Restrictions

To avoid complex number values, the base b is positive: $b > 0$. (1)

Indeed, if the base is negative, then the value of the exponential function is a complex number.

For example: $(-4)^{\frac{1}{2}} = \sqrt{-4} = \pm 2i$. (complex numbers)

In order the exponential function to be a one-to-one function, the base must not be one: $b \neq 1$ (2)

Indeed, if the base is one, then $1^x = 1$, which is not a one-to-one function and does not have an inverse function. If $b=1$ then $b^x = 1^x$

not a function

So, in conclusion the restrictions are:

$b > 0, b \neq 1$ or $b \in (0,1) \cup (1, \infty)$ $y = 1$

Ex 2. Determine if the exponential function is well defined (satisfies the restrictions).

a) $y = (-2)^x$ (incorrect) $b = -2 < 0$

b) $y = -2^x = -(2^x)$ (ok)

c) $y = -1^x = -(1^x)$ (incorrect) $b = 1$

d) $y = \frac{1}{(-2)^x} = [(-2)^{-1}]^x = (-\frac{1}{2})^x$ (incorrect) $b = -\frac{1}{2} < 0$

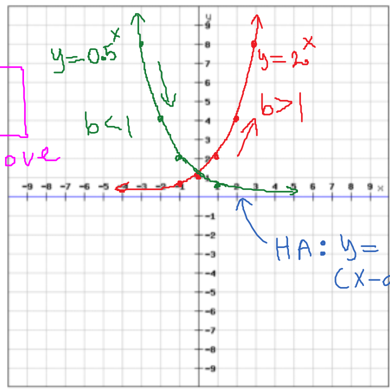
D The Graph of the Exponential Function

Ex 3. Graph on the same grid.

a) $y = 2^x$

b) $y = \left(\frac{1}{2}\right)^x = 2^{-x} = 0.5^x$

$\boxed{b^x > 0}$ (graph is above the x-axis)



HA: $y = 0$ (x-axis)

$y = 2^x$
$y = 2^{-x}$ reflection in the y-axis

Note. $y = \left(\frac{1}{2}\right)^x = 2^{-x}$, so the graphs are symmetric with respect to the y-axis.

E Characteristics of the Exponential Function

Ex 4. Use the graphs obtained at Ex 3 to find the following characteristics of the exponential function.

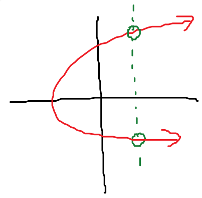
$\boxed{y = b^x}$

- Domain: $\mathbb{R}$
- Range: $(0, \infty)$
- x-intercepts none
- y-intercepts: $(0, 1)$
- Increasing/Decreasing: increasing if $b > 1$; decreasing if $b < 1$
- Horizontal Asymptotes: $y = 0$ (x-axis)
- Vertical Asymptotes: none
- Continuity: yes
- One-to-one: yes
- Key Points: $(0,1)$ $(1,b)$ $\left(-1, \frac{1}{b}\right)$

F Transformations

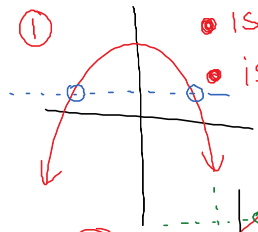
$$g(x) = A\,b^{B(x-C)} + D$$

Note. The equation of the horizontal asymptote is: $y = D$



not a function
does not pass the Vertical Line Test

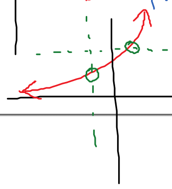
1



- is a function
- is not one-to-one

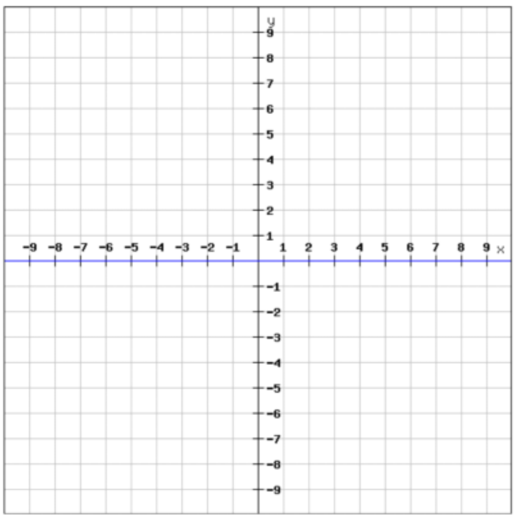
does not pass the Horizontal Line Test

3

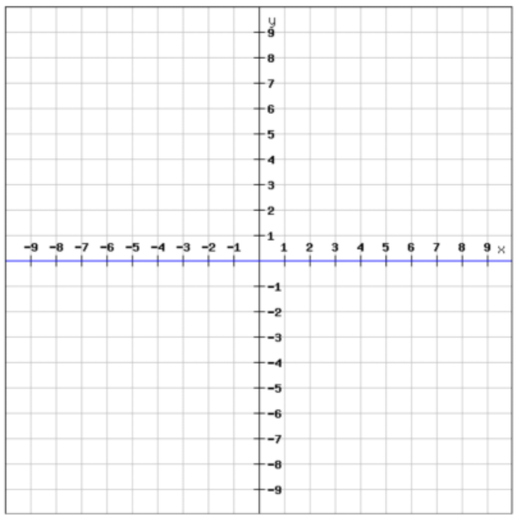


- is a function
- is one-to-one

Ex 5. Use transformations to graph the following exponential functions.



a) $y = -3^{-x+1} + 9$



b) $y = -4 + 2 \cdot 0.5^{x+2}$

G One-to-one property

The exponential function is a one-to-one function. Therefore:

$$b^x = b^y \Leftrightarrow x = y$$