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7.4 Proving Trigonometric Identities

A Equations versus Identities

An equation is a mathematical statement which is true if the variable has only specific values. To solve an equation means to find the set of values of the variable that satisfy the equation. → roots (solutions)

An identity is a mathematical statement which is true for any values of the variable. To prove an identity means to start with a side and apply algebraic rules until the other side is reached.

Ex 1. Classify the following expressions as identity or equation:

a) $x^2 - 4 = 0$ (equation) $x = \pm 2$ (solutions)

b) $(x+2)(x-2) = x^2 - 4$ (identity) $x \in \mathbb{R}$

B Pythagorean Identities

The following identities are called Pythagorean Identities:

$$\sin^2 x + \cos^2 x = 1 \quad (1)$$

$$1 + \tan^2 x = \sec^2 x \quad (7)$$

$$1 + \cot^2 x = \csc^2 x \quad (8)$$

C Identities based on Definitions

The following definitions might be used when proving trigonometric identities:

$$\tan x = \frac{\sin x}{\cos x} \quad (2) \qquad \cot x = \frac{\cos x}{\sin x} \quad (4) \qquad \cot x = \frac{1}{\tan x} \quad (6)$$

$$\sec x = \frac{1}{\cos x} \quad (3) \qquad \csc x = \frac{1}{\sin x} \quad (5)$$

Ex 2. Prove the following identity

$1 - \cos^2 x = \cos^2 x \tan^2 x$

$RS = \cos^2 x \tan^2 x$ ← $\tan x = \frac{\sin x}{\cos x}$

$= \cos^2 x \frac{\sin^2 x}{\cos^2 x}$

$= \sin^2 x$ ← $\sin^2 x + \cos^2 x = 1$

$= 1 - \cos^2 x$

$= LS$

Ex 3. Prove the following identity

$\frac{\sin x}{1 - \cos x} = \frac{1 + \cos x}{\sin x}$

$\cos x \neq 1$, $\sin x \neq 0$

$x \neq k(2\pi)$ and $x \neq k\pi$

$\boxed{\therefore x \neq k\pi}$

$\sin^2 x = (1 + \cos x)(1 - \cos x)$

$\sin^2 x = 1 - \cos^2 x$

true

Ex 4. Prove the following identity

$\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x} = 2\sec^2 x$

$\sin x \neq 1, -1$

$\boxed{x \neq \frac{\pi}{2} + k\pi}$

$LS = \frac{1}{1 + \sin x} + \frac{1}{1 - \sin x}$

$= \frac{1 - \sin x + 1 + \sin x}{(1 + \sin x)(1 - \sin x)}$

$= \frac{2}{1 - \sin^2 x}$ ← $\sin^2 x + \cos^2 x = 1$

$= \frac{2}{\cos^2 x}$ ← $\sec x = \frac{1}{\cos x}$

$= 2\sec^2 x$

$= RS$

Ex 5. Prove the following identity

$\frac{1 + \cos 2x}{\sin 2x} = \cot x$

$LS = \frac{1 + \cos 2x}{\sin 2x}$ ← $\cos 2x = 2\cos^2 x - 1$, $\sin 2x = 2\sin x \cos x$

$= \frac{1 + 2\cos^2 x - 1}{2 \sin x \cos x}$

$= \frac{\cos x}{\sin x}$ ← $\cot x = \frac{\cos x}{\sin x}$

$= \cot x$

$= RS$

Reading: Nelson Textbook, Pages 412-416

Homework: Nelson Textbook, Page 417: #8, 9, 10, 11, 16

More Trigonometric Identities

Part A

1. $\tan x \cos x = \sin x$
2. $\cot x \sec x = \csc x$
3. $\sin x \cot x = \cos x$
4. $\tan x \csc x = \sec x$
5. $\sin x = \dfrac{\tan x}{\sec x}$
6. $\dfrac{\cot x}{\csc x} = \cos x$
7. $(1+\sin x)\csc x = 1+\csc x$
8. $(1+\csc x)\sin x = 1+\sin x$
9. $(\sec x - 1)\cos x = 1-\cos x$
10. $\sin x \sec x \cot x = 1$
11. $\dfrac{1-\tan x}{1-\cot x} = -\tan x$
12. $\cot x = \dfrac{1+\cot x}{1+\tan x}$
13. $\sin x \tan x + \sec x = \dfrac{\sin^2 x + 1}{\cos x}$
14. $\dfrac{1+\cos x}{1-\cos x} = \dfrac{1+\sec x}{\sec x - 1}$
15. $\dfrac{1+\sin x}{1-\sin x} = \dfrac{\csc x + 1}{\csc x - 1}$
16. $\dfrac{1+\tan x}{1+\cot x} = \dfrac{1-\tan x}{\cot x - 1}$
17. $\dfrac{1+\sin x}{1+\csc x} = \sin x$
18. $\dfrac{\sin x + \tan x}{\cos x + 1} = \tan x$
19. $\sin^2 x \cot^2 x = 1-\sin^2 x$
20. $\csc^2 x - 1 = \csc^2 x \cos^2 x$
21. $\sin^2 x = \dfrac{\tan^2 x}{1+\tan^2 x}$
22. $\dfrac{\sin x + \cos x \cot x}{\cot x} = \sec x$
23. $\dfrac{\cos x \tan x}{\csc x} = 1-\cos^2 x$
24. $\dfrac{\cos x}{1+\sin x} + \dfrac{\cos x}{1-\sin x} = 2\sec x$
25. $\dfrac{\sin x + \cos x}{\csc x + \sec x} = \sin x \cos x$
26. $\dfrac{\tan x}{\sec x + 1} = \dfrac{\sec x - 1}{\tan x}$
27. $\dfrac{1}{1+\sin x} + \dfrac{1}{1-\sin x} = 2\sec^2 x$
28. $(1+\cot^2 x)\tan^2 x = \sec^2 x$
29. $(1-\cos^2 x)(1+\tan^2 x) = \tan^2 x$
30. $\tan x + \cot x = \sec x \csc x$
31. $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$
32. $\sec^2 x + \csc^2 x = (\tan x + \cot x)^2$
33. $\sin^2 x = \cos x(\sec x - \cos x)$
34. $\tan x + \tan^3 x = \dfrac{\sec^2 x}{\cot x}$
35. $\dfrac{1+\csc x}{\cot x} - \sec x = \tan x$
36. $\dfrac{(1-\cos^2 x)(\sec^2 x - 1)}{\cos^2 x} = \tan^4 x$
37. $(1+\sin x)\sec x = (1+\csc x)\tan x$
38. $\csc x - \dfrac{\cot x}{\sec x} = \sin x$
39. $\cos^2 x = (\csc x - \sin x)\sin x$
40. $\sec^2 x - 1 = (\sin x \sec x)^2$
41. $\cos^2 x = \dfrac{\cot^2 x}{1+\cot^2 x}$
42. $\dfrac{\sin x}{1-\cos x} + \dfrac{\sin x}{1+\cos x} = 2\csc x$
43. $\sin x \cot^2 x + \cos x \tan^2 x = \dfrac{\sin^3 x + \cos^3 x}{\sin x \cos x}$
44. $\dfrac{\tan^2 x + 1}{\cot^2 x + 1} = \dfrac{1-\cos^2 x}{\cos^2 x}$
45. $2\cos^2 x - 1 = \cos^4 x - \dfrac{1}{\csc^4 x}$
46. $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$
47. $\dfrac{\csc x}{\sec^2 x} = \csc x - \sin x$
48. $\dfrac{\sin x + \tan x}{1+\cos x} = \tan x$
49. $\dfrac{\cos x}{1-\sin x} + \dfrac{\cos x}{1+\sin x} = 2\sec x$
50. $\dfrac{1+\sin x + \cos x}{1-\sin x + \cos x} = \dfrac{1+\sin x}{\cos x}$
51. $\dfrac{1-\sin x}{1+\sin x} = (\tan x - \sec x)^2$

Part B

1. $1+\sin 2x=(\sin x+\cos x)^2$
2. $\sin 2x = 2\cot x \sin^2 x$
3. $\cos 2x=\dfrac{1-\tan^2 x}{1+\tan^2 x}$
4. $\sec^2 x=\dfrac{2}{1+\cos 2x}$
5. $\dfrac{1-\cos 2x}{2}=\sin^2 x$
6. $\dfrac{\sin^2 x+\cos^2 x}{\sin^2 x-\cos^2 x}=-\sec 2x$
7. $\dfrac{(\sin x+\cos x)^2}{\sin 2x}=1+\csc 2x$
8. $\dfrac{\sin 2x}{1-\cos 2x}=\dfrac{1}{\tan x}$
9. $\dfrac{\cos 2x}{\sin 2x+1}=\dfrac{1-\tan x}{1+\tan x}$
10. $\cot\dfrac{\alpha}{2}=\dfrac{1+\cos\alpha}{\sin\alpha}$
11. $\tan\dfrac{\alpha+\beta}{2}=\dfrac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}$
12. $\cos(a+b)\cos(a-b)=\cos^2 b-\sin^2 a$
13. $\sin^2\alpha\cos^2\alpha=\dfrac{1-\cos 4\alpha}{8}$
14. $\sin x\sin y=\dfrac{\cos(x-y)-\cos(x+y)}{2}$
15. $\cos x\cos y=\dfrac{\cos(x-y)+\cos(x+y)}{2}$
16. $\sin x\cos y=\dfrac{\sin(x-y)+\sin(x+y)}{2}$
17. $\sin x+\sin y=2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$
18. $\cos x+\cos y=2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$
19. $\cos x-\cos y=-2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$
20. $\sin x-\sin y=2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$
21. $\cos u+\sin v=\left(\cos\dfrac{u-v}{2}-\sin\dfrac{u-v}{2}\right)\left(\cos\dfrac{u+v}{2}+\sin\dfrac{u+v}{2}\right)$
22. $\cos u-\sin v=\left(\cos\dfrac{u+v}{2}-\sin\dfrac{u+v}{2}\right)\left(\cos\dfrac{u-v}{2}+\sin\dfrac{u-v}{2}\right)$
23. $\cos(a-b)\cos(t+u)-\cos(a+b)\cos(t-u)=\sin(u+a)\sin(b-t)-\sin(u-a)\sin(b+t)$

$$RS=\frac{\cancel{\cos x\cos y}+\sin x\sin y-(\cancel{\cos x\cos y}-\sin x\sin y)}{2}=LS$$

$$LS=\sin\Big(\overbrace{\frac{x+y}{2}+\frac{x-y}{2}}^{x}\Big)+\sin\Big(\overbrace{\frac{x+y}{2}-\frac{x-y}{2}}^{y}\Big)=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\frac{x}{2}+\cancel{\frac{y}{2}}+\frac{x}{2}-\cancel{\frac{y}{2}}$$

$$\cancel{\frac{x}{2}}+\frac{y}{2}-\cancel{\frac{x}{2}}+\frac{y}{2}=y$$

Part C

1. If $x+y+z=\pi$ prove that $\sin 2x+\sin 2y+\sin 2z=4\sin x\sin y\sin z$.

2. If $x+y+z=\pi$ prove that $\tan x+\tan y+\tan z=\tan x\tan y\tan z$.

3. If $x+y+z=\pi/2$ prove that $\cot x+\cot y+\cot z=\cot x\cot y\cot z$.

Part D

1. $$\sin\alpha+\sin(\alpha+\varepsilon)+\sin(\alpha+2\varepsilon)+\ldots+\sin(\alpha+n\varepsilon)=\frac{\sin\dfrac{(n+1)\varepsilon}{2}\sin\left(\alpha+\dfrac{n\varepsilon}{2}\right)}{\sin\dfrac{\varepsilon}{2}}$$

2. $$\cos\alpha+\cos(\alpha+\varepsilon)+\cos(\alpha+2\varepsilon)+\ldots+\cos(\alpha+n\varepsilon)=\frac{\sin\dfrac{(n+1)\varepsilon}{2}\cos\left(\alpha+\dfrac{n\varepsilon}{2}\right)}{\sin\dfrac{\varepsilon}{2}}$$

Hint: Multiply both sides by $\sin\dfrac{\varepsilon}{2}$ and use the identities **14-20** from **Part B**.

7 $1+\tan^2 x=\sec^2 x$

$LS=1+\tan^2 x$; $\tan x=\dfrac{\sin x}{\cos x}$

$=1+\dfrac{\sin^2 x}{\cos^2 x}$

$=\dfrac{\cos^2 x+\sin^2 x}{\cos^2 x}$; $\sin^2 x+\cos^2 x=1$

$=\dfrac{1}{\cos^2 x}$; $\sec x=\dfrac{1}{\cos x}$

$=\sec^2 x$

$=RS$

10

$$RS=\frac{1+\cos\alpha}{\sin\alpha}=\frac{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}{\sin\left(2\,\frac{\alpha}{2}\right)}$$

$$= \frac{\cancel{1} + \cancel{2}\cos^2\frac{\alpha}{2} - \cancel{1}}{\cancel{2}\sin\frac{\alpha}{2}\cancel{\cos\frac{\alpha}{2}}} = \frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} = \cot\frac{\alpha}{2} = LS$$