Nelson 7.3 Double Angle Formulas

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Note

$\sin 3x = \sin(3x)$

$\sin^3 x = (\sin x)^3$

② $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\alpha = \beta = x$

$\cos 2x = \cos^2 x - \sin^2 x$

③ $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\alpha = \beta = x$ $\quad \tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

7.3 Double Angle Formulas

A Double-Angle Identities

The double-angle identities are:

$$\sin 2x = 2\sin x\cos x \quad ①$$
$$\cos 2x = \cos^2 x - \sin^2 x \quad ②$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} \quad ③$$

Proof for ①

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\alpha = \beta = x$

$\sin 2x = 2\sin x\cos x$

Ex 1. Given that $\boxed{\sin x = -\frac{1}{3}}$ and that the angle x is in the third quadrant, find $\sin 2x$, $\cos 2x$ and $\tan 2x$.

$x \in Q\#3 \Rightarrow \cos x < 0$

$\sin^2 x + \cos^2 x = 1 \Rightarrow \left(-\frac{1}{3}\right)^2 + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \frac{1}{9} = \frac{8}{9}$

$\boxed{\cos x = -\frac{2\sqrt{2}}{3}}$

$\sin 2x = 2\sin x\cos x = 2\left(-\frac{1}{3}\right)\left(\frac{-2\sqrt{2}}{3}\right) = \frac{4\sqrt{2}}{9}$

$\cos 2x = \cos^2 x - \sin^2 x = \frac{8}{9} - \frac{1}{9} = \frac{7}{9}$

$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{4\sqrt{2}}{7}$

B Alternate Formulas

The double angle cosine has the following alternate formulas:

$$\cos 2x = \cos^2 x - \sin^2 x \quad ②$$
$$\cos 2x = 2\cos^2 x - 1 \quad ④$$
$$\cos 2x = 1 - 2\sin^2 x \quad ⑤$$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$\cos^2 x = 1 - \sin^2 x$

Ex 2. Use the double angle formulas and the fundamental trigonometric identity $\sin^2 x + \cos^2 x = 1$ to prove the alternate formulas.

$\cos 2x = \cos^2 x - (1 - \cos^2 x)$

$= 2\cos^2 x - 1$ ④

$\cos 2x = 1 - \sin^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$ ⑤

C Half-Angle Identities (optional to memorize)

The half-angle identities are:

$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$$
$$\cos^2\frac{x}{2} = \frac{1+\cos x}{2}$$
$$\tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x}$$

or:

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$
$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

Ex 3. Find the exact value of: $\frac{\pi}{8}$ in the first quadrant $\Rightarrow \sin\frac{\pi}{8} > 0$

a) $\sin\dfrac{\pi}{8}$

$\cos 2x = 1 - 2\sin^2 x \Rightarrow \cos\left(2\cdot\frac{\pi}{8}\right) = 1 - 2\sin^2\frac{\pi}{8}$

$2\sin^2\frac{\pi}{8} = 1 - \cos\frac{\pi}{4} = 1 - \frac{\sqrt{2}}{2} \Rightarrow \sin\frac{\pi}{8} = +\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$

$\therefore \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$

b) $\cos\dfrac{\pi}{12}$

Method #1 $\quad \cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cdot$

Method #2 $\quad \cos 2\alpha = 2\cos^2\alpha - 1 \Rightarrow \cos\left(2\frac{\pi}{12}\right) = 2\cos^2\frac{\pi}{12} - 1$

$2\cos^2\frac{\pi}{12} = 1 + \cos\frac{\pi}{6} = 1 + \frac{\sqrt{3}}{2} \Rightarrow \cos\frac{\pi}{12} = +\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$ $\quad \therefore \cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$

c) $\tan\dfrac{5\pi}{8} < 0$ because $\frac{5\pi}{8} \in Q\#2$

Method #1 $\quad \tan\frac{5\pi}{8} = \frac{\sin(5\pi/8)}{\cos(5\pi/8)} = \cdots\cdots$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{\quad}{2}}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

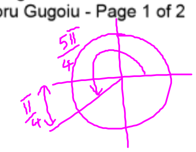
c) $\tan\frac{5\pi}{8} < 0$ because $\frac{5\pi}{8} \in Q\#2$

Method #1 $\quad \tan\frac{5\pi}{8} = \frac{\sin(5\pi/8)}{\cos(5\pi/8)} = \ldots\ldots$

Method #2

$\tan 2x = \frac{2\tan x}{1-\tan^2 x} \Rightarrow \tan\left(2\,\frac{5\pi}{8}\right) = \frac{2\tan(5\pi/8)}{1-\tan^2(5\pi/8)}$ $\quad$ $x = \frac{5\pi}{8}$

$t = \tan\frac{5\pi}{8} = ?$



$\underbrace{\tan\frac{5\pi}{4}}_{\underbrace{\tan\frac{\pi}{4}}_{1}} = \frac{2t}{1-t^2} = 1 \Rightarrow 2t = 1-t^2$

$t^2+2t-1=0$

$t = \frac{-2\pm\sqrt{4+4}}{2}$

$t = -1\pm\sqrt{2}$

$\therefore \tan\frac{5\pi}{8} = -1-\sqrt{2}$

D Triple-Angle Identities

The triple-angle identities are:

$$\sin 3x = 3\sin x - 4\sin^3 x \quad (a)$$
$$\cos 3x = 4\cos^3 x - 3\cos x \quad (b)$$
$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \quad (c)$$

Ex 4. Prove the triple-angle identities.

(a) $LS = \sin 3x = \sin(x+2x)$
$= \sin x \cos 2x + \cos x \sin 2x = \sin x(1-2\sin^2 x) + \cos x(2\sin x \cos x)$
$= \sin x - 2\sin^3 x + 2\sin x \underbrace{\cos^2 x}_{(1-\sin^2 x)}$
$= 3\sin x - 4\sin^3 x = RS$

(b) $\cos 3x = \cos(x+2x) = \cos x \cos 2x - \sin x \sin 2x$
$= \cos x(2\cos^2 x - 1) - (\sin x)(2\sin x \cos x)$
$= \cos x(2\cos^2 x - 1) - 2\cos x \underbrace{\sin^2 x}_{(1-\cos^2 x)}$
$= 4\cos^3 x - 3\cos x = RS$

(c) $\tan 3x = \tan(x+2x) = \dfrac{\tan x + \tan 2x}{1 - \tan x \tan 2x}$
$= \dfrac{\tan x + \dfrac{2\tan x}{1-\tan^2 x}}{1 - \tan x \dfrac{2\tan x}{1-\tan^2 x}} = \dfrac{\dfrac{\tan x - \tan^3 x + 2\tan x}{\cancel{1-\tan^2 x}}}{\dfrac{1-\tan^2 x - 2\tan^2 x}{\cancel{1-\tan^2 x}}}$
$= \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} = RS$

E Half-Angle Tangent Formulas (Weierstrass *t*-substitution)

The following formulas are called half-angle tangent formulas or Weierstrass *t*-substitution:

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \quad (a)$$
$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} \quad (b)$$
$$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} \quad (c)$$

$RS = \dfrac{2\tan(x/2)}{1-\tan^2(x/2)}$

$= \dfrac{2\dfrac{\sin(x/2)}{\cos(x/2)}}{1-\dfrac{\sin^2(x/2)}{\cos^2(x/2)}} = \dfrac{2\sin\frac{x}{2}}{\cos\frac{x}{2}} \cdot \dfrac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}$

Ex 5. Prove the half-angle tangent (Weierstrass *t*-substitution) formulas.

(a) $RS = \dfrac{2\tan(x/2)}{1+\tan^2(x/2)} = \dfrac{2\dfrac{\sin(x/2)}{\cos(x/2)}}{1+\dfrac{\sin^2(x/2)}{\cos^2(x/2)}}$

$= \dfrac{\dfrac{2\sin(x/2)}{\cos(x/2)}}{\dfrac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \dfrac{2\sin\frac{x}{2}}{\cancel{\cos\frac{x}{2}}} \cdot \dfrac{\cos^{\cancel{2}}\frac{x}{2}}{1}$ ($1 = \cos^2\frac{x}{2}+\sin^2\frac{x}{2}$)

$= 2\sin\frac{x}{2}\cos\frac{x}{2} = \sin\left(2\cdot\frac{x}{2}\right) = \sin x = LS$

(b) $RS = \dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$

$= \dfrac{1-\dfrac{\sin^2(x/2)}{\cos^2(x/2)}}{1+\dfrac{\sin^2(x/2)}{\cos^2(x/2)}} = \dfrac{\cos^2(x/2)-\sin^2(x/2)}{\cos^2(x/2)+\sin^2(x/2)}$

$= \dfrac{\cos(2\cdot\frac{x}{2})}{1} = \cos x = LS$

$= \dfrac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}} = \dfrac{\sin x}{\cos x} = \tan x = LS$

Reading: Nelson Textbook, Pages 402-406
Homework: Nelson Textbook, Page 407: #1abd, 2ae, 4, 5, 9, 10, 11, 12, 13, 15