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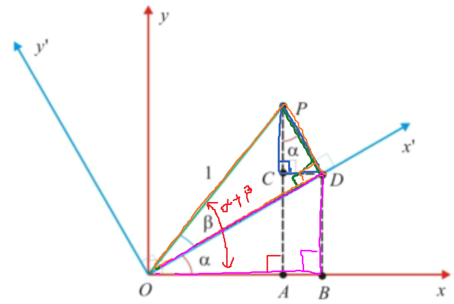
7.2 Compound Angle Formulas

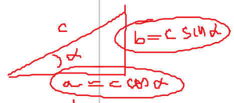
A Addition Formulas

The following relations are trigonometric identities called addition formulas:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \quad ①$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \quad ②$$
$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \quad ③$$

Proof:





$\sin\alpha=\frac{b}{c}$

$\cos\alpha=\frac{a}{c}$

$\sin(\alpha+\beta)=\frac{AP}{1}=AC+CP$

CP: PD cos α, (1) sin β

AC: BD, OD sin α, (1) cos β

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

Ex 1. Use the addition formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ to prove the addition formula for $\tan(\alpha+\beta)$.

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

$$=\frac{\frac{\sin\alpha\,\cancel{\cos\beta}}{\cos\alpha\,\cancel{\cos\beta}}+\frac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\frac{\cancel{\cos\alpha\cos\beta}}{\cancel{\cos\alpha\cos\beta}}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \quad ③$$

B Subtraction Formulas

The following relations are trigonometric identities called subtraction formulas:

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \quad ④$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \quad ⑤$$
$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \quad ⑥$$

$\sin(-\alpha)=-\sin\alpha$

$\cos(-\alpha)=\cos\alpha$

Ex 2. Replace in the addition formulas β by $-\beta$, then use the symmetry of the sine and cosine functions and the addition formula to prove the subtraction formulas.

$\Rightarrow \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\beta\to-\beta$

$\sin[\alpha+(-\beta)]=\sin\alpha\underbrace{\cos(-\beta)}_{\cos\beta}+\cos\alpha\underbrace{\sin(-\beta)}_{-\sin\beta}$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \quad ④$

Ex 3. Use $\sin(\pi/2)=1$ and $\cos(\pi/2)=0$ to prove the following formulas:

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \quad ① ✓$$
$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \quad ②$$
$$\div \quad \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta=\frac{1}{\tan\theta}=\frac{1}{\frac{\sin\theta}{\cos\theta}}=\frac{\cos\theta}{\sin\theta}$$

① $\sin\left(\frac{\pi}{2}-\theta\right)=\underbrace{\sin\frac{\pi}{2}}_{1}\cos\theta-\cancel{\cos\frac{\pi}{2}\sin\theta}$
$=\cos\theta$

② $\cos\left(\frac{\pi}{2}-\theta\right)=\cancel{\cos\frac{\pi}{2}\cos\theta}+\underbrace{\sin\frac{\pi}{2}}_{1}\sin\theta$
$=\sin\theta$

Ex 4. Use $\sin(\pi/2)=1$ and $\cos(\pi/2)=0$ to prove the following formulas: $$\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$$ $$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$$ $$\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$$	
Ex 5. Use $\sin(\pi)=0$ and $\cos(\pi)=-1$ to prove the following formulas: $$\sin(\pi-\theta)=\sin\theta$$ $$\cos(\pi-\theta)=-\cos\theta$$ $$\tan(\pi-\theta)=-\tan\theta$$	
Ex 6. Use $\sin(\pi)=0$ and $\cos(\pi)=-1$ to prove the following formulas: $$\sin(\pi+\theta)=-\sin\theta$$ $$\cos(\pi+\theta)=-\cos\theta$$ $$\tan(\pi+\theta)=\tan\theta$$	
Ex 7. Use $\sin(3\pi/2)=-1$ and $\cos(3\pi/2)=0$ to prove the following formulas: $$\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta$$ $$\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta$$ $$\tan\left(\frac{3\pi}{2}-\theta\right)=\cot\theta$$	
Ex 8. Use $\sin(3\pi/2)=-1$ and $\cos(3\pi/2)=0$ to prove the following formulas: $$\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta$$ $$\cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta$$ $$\tan\left(\frac{3\pi}{2}+\theta\right)=-\cot\theta$$	
Ex 9. Use $\sin(\pi)=0$ and $\cos(\pi)=-1$ to prove the following formulas: $$\sin(2\pi-\theta)=-\sin\theta$$ $$\cos(2\pi-\theta)=\cos\theta$$ $$\tan(2\pi-\theta)=-\tan\theta$$	

with radicals — no decimals

Ex 10. Find the exact value of $\sin\frac{5\pi}{12}$.

$\frac{\pi}{3}$ $\frac{\pi}{4}$ $\frac{\pi}{6}$ $\qquad \frac{\pi}{4}+\frac{\pi}{6}=\frac{10\pi}{24}=\frac{5\pi}{12}$

$\sin\frac{5\pi}{12}=\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$

$=\sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6}$

$=\frac{\sqrt2}{2}\,\frac{\sqrt3}{2}+\frac{\sqrt2}{2}\,\frac{1}{2}=\frac{\sqrt2(\sqrt3+1)}{4}$

(exact value)

Ex 11. Find the exact value of $\cos\frac{\pi}{12}$.

$\cos\frac{\pi}{12}=\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$

$=\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}$

$=\frac{1}{2}\,\frac{\sqrt2}{2}+\frac{\sqrt3}{2}\,\frac{\sqrt2}{2}$

$=\frac{\sqrt2(\sqrt3+1)}{4}\Rightarrow \sin\frac{5\pi}{12}=\cos\frac{\pi}{12}$

$\frac{\pi}{2}-\frac{5\pi}{12}$

$\sin\alpha=\cos\left(\frac{\pi}{2}-\alpha\right)$

Ex 12. Find the exact value of $\tan\frac{7\pi}{12}$.

$\tan\frac{7\pi}{12}=\tan\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$

$=\frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{3}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{3}}$

$=\frac{1+\sqrt3}{1-\sqrt3}\cdot\frac{1+\sqrt3}{1+\sqrt3}=\frac{4+2\sqrt3}{-2}$

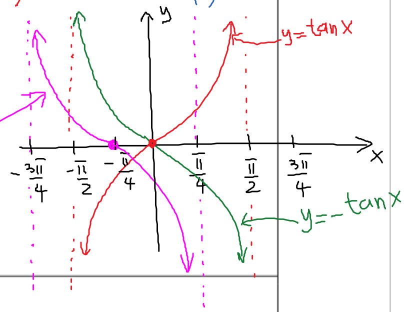
$=-2-\sqrt3$

Ex 13. Graph $f(x)=\frac{\sin x+\cos x}{\sin x-\cos x}$.

$=\frac{\sin x\,\overbrace{\frac{\sqrt2}{2}}^{\cos\pi/4}+\overbrace{\frac{\sqrt2}{2}}^{\sin\pi/4}\cos x}{-\left(\cos x\,\frac{\sqrt2}{2}-\frac{\sqrt2}{2}\sin x\right)}=-\frac{\sin\left(x+\frac{\pi}{4}\right)}{\cos\left(x+\frac{\pi}{4}\right)}$

$\boxed{f(x)=-\tan\left(x+\frac{\pi}{4}\right)}$

$y=\frac{\sin x+\cos x}{\sin x-\cos x}$



Ex 14. Prove that:

$$\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

$\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}=\cdots$

$=\frac{1}{\tan(\alpha+\beta)}$

$=\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}=\frac{1-\frac{1}{\cot\alpha}\,\frac{1}{\cot\beta}}{\frac{1}{\cot\alpha}+\frac{1}{\cot\beta}}\cdot\frac{\cot\alpha\cot\beta}{\cot\alpha\cot\beta}$

Ex 15. Prove that:

$$\csc(\alpha+\beta)=\frac{\csc\alpha\csc\beta}{\cot\alpha+\cot\beta}$$

$\csc(\alpha+\beta)=\frac{1}{\sin(\alpha+\beta)}$

$=\frac{1}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}\cdot\frac{\frac{1}{\sin\alpha\sin\beta}}{\frac{1}{\sin\alpha\sin\beta}}$

$=\frac{\csc\alpha\csc\beta}{\cot\beta+\cot\alpha}$

Reading: Nelson Textbook, Pages 394-399
Homework: Nelson Textbook, Page 400: #2a, 5abc, 8ade, 9bcf, 12b, 16, 17

$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$

$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$

$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$