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Thursday, November 5, 2020 7:39 AM

6.2 Radian Measure and Angles on the Cartesian Plane

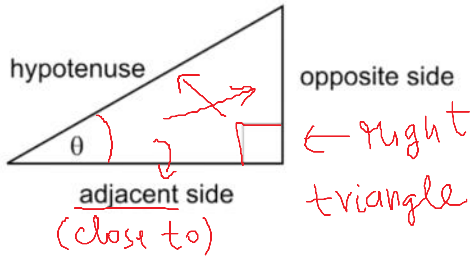
A Trigonometric Ratios

The trigonometric ratios are defined by:

$$\sin\theta = \frac{opposite}{hypotenuse}$$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$

$$\tan\theta = \frac{opposite}{adjacent}$$



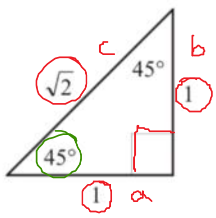
B Special Triangles

$c^2 = a^2 + b^2$

Pythagorean Theorem

$c^2 = 1^2 + 1^2 = 2$

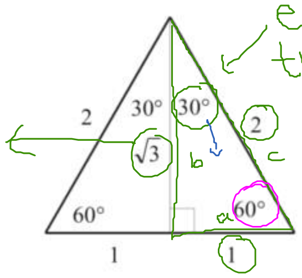
$c = \sqrt{2}$



$2^2 = 1^2 + b^2$

$3 = b^2$

$b = \sqrt{3}$



Ex 1. Use the special triangles to find the values of the following trigonometric ratios.

a) $\sin 45° = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

b) $\cos 45° = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

c) $\tan 45° = \tan\frac{\pi}{4} = \frac{1}{1} = 1$

d) $\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2}$

e) $\cos 30° = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

f) $\tan 30° = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

g) $\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

h) $\cos 60° = \cos\frac{\pi}{3} = \frac{1}{2}$

i) $\tan 60° = \tan\frac{\pi}{3} = \frac{\sqrt{3}}{1} = \sqrt{3}$

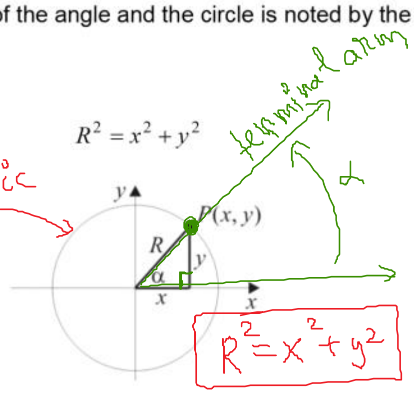
C Trigonometric Functions

Consider a circle of radius R and an angle α in standard position. The intersection between the terminal arm of the angle and the circle is noted by the point $P(x, y)$.

Notes:

$$R^2 = x^2 + y^2$$

trigonometric circle



$\boxed{R^2 = x^2 + y^2}$

The trigonometric functions are defined by:

$$\sin(\alpha) = \sin\alpha = \frac{y}{R}$$

$$\cos(\alpha) = \cos\alpha = \frac{x}{R}$$

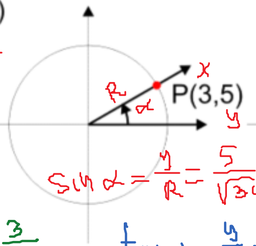
÷

$$\tan(\alpha) = \tan\alpha = \frac{y}{x}$$

Note.

$$\boxed{\tan\alpha = \frac{\sin\alpha}{\cos\alpha}}$$

Ex 2. For each case, find the value of sine, cosine, and tangent functions.

a) 

P(3,5)

$R^2 = 3^2 + 5^2$

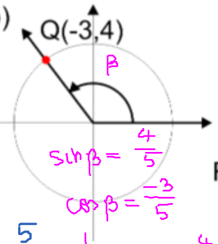
$R^2 = 9 + 25$

$R = \sqrt{34}$

$\sin\alpha = \frac{y}{R} = \frac{5}{\sqrt{34}}$

$\cos\alpha = \frac{x}{R} = \frac{3}{\sqrt{34}}$

$\tan\alpha = \frac{y}{x} = \frac{5}{3}$

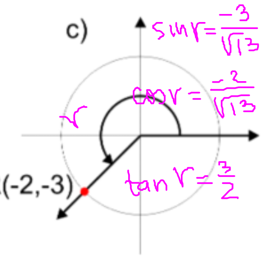
b) 

Q(-3,4)

$\sin\beta = \frac{4}{5}$

$\cos\beta = \frac{-3}{5}$

$\tan\beta = -\frac{4}{3}$

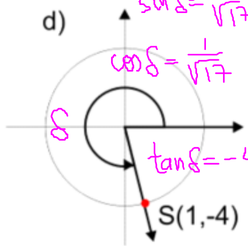
c) 

R(-2,-3)

$\sin\gamma = \frac{-3}{\sqrt{13}}$

$\cos\gamma = \frac{-2}{\sqrt{13}}$

$\tan\gamma = \frac{3}{2}$

d) 

S(1,-4)

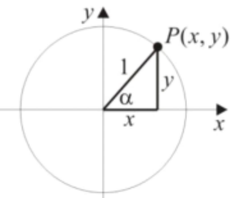
$\sin\delta = \frac{-4}{\sqrt{17}}$

$\cos\delta = \frac{1}{\sqrt{17}}$

$\tan\delta = -4$

D Unit Circle

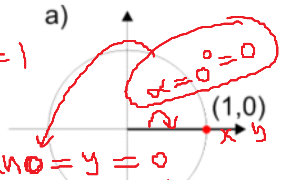
$R = 1$



If the circle has a radius $R = 1$ (unit circle) then the trigonometric functions are defined by:

$$\sin(\alpha) = \sin\alpha = y$$
$$\cos(\alpha) = \cos\alpha = x$$
$$\tan(\alpha) = \tan\alpha = \frac{y}{x}$$

Ex 3. For each case, find the value of sine, cosine, and tangent functions.

a) 

(1,0)

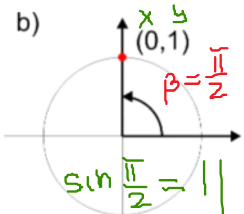
$R = 1$

$\alpha = 0° = 0$

$\sin 0 = y = 0$

$\cos 0 = x = 1$

$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0$

b) 

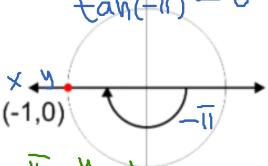
(0,1)

$\beta = \frac{\pi}{2}$

$\sin\frac{\pi}{2} = 1$

$\cos\frac{\pi}{2} = 0$

$\tan\frac{\pi}{2} = \frac{y}{x} = \frac{1}{0} = \text{und}$

c) 

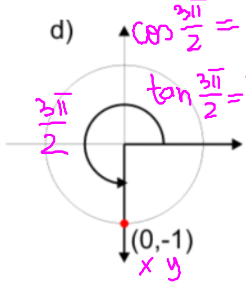
(-1,0)

$-\pi$

$\sin(-\pi) = 0$

$\cos(-\pi) = -1$

$\tan(-\pi) = 0$

d) 

(0,-1)

$\frac{3\pi}{2}$

$\sin\frac{3\pi}{2} = -1$

$\cos\frac{3\pi}{2} = 0$

$\tan\frac{3\pi}{2} = \frac{-1}{0} = \text{und}$

E Fundamental Trigonometric Identity

For any angle α the following identity is true:

$$\sin^2\alpha + \cos^2\alpha = 1$$

Proof:

F Domain and Range

The domain for the sine and cosine functions is the real numbers set. The range for the sine and cosine functions is $[-1,1]$.

Proof:

The domain for the tangent function is $\{\alpha \in R \mid \alpha \neq (2k+1)\frac{\pi}{2}\}$ and the range is the real numbers set.

$\frac{\pi}{2} + k\pi = \frac{(2k+1)\pi}{2}$

Proof:

G Sign of Trigonometric Functions

The sign of sine functions is the sign of the coordinate y.

The sign of cosine functions is the sign of the coordinate x.

The sign of tangent functions is the sign of the ratio y/x.

Ex 4. The sine of a given angle α is equal to $-\frac{2}{3}$.

Find $\cos\alpha$ and $\tan\alpha$.

$\sin\alpha = \frac{-2}{3} = \frac{y}{R} \Rightarrow$

$y = -2$

$R = 3$

$x^2 + y^2 = R^2$

$x^2 + 4 = 9$

$x^2 = 5$

$x = \pm\sqrt{5}$

$x = \sqrt{5}$

$\cos\alpha = \frac{x}{R} = \frac{\sqrt{5}}{3}$

$\tan\alpha = \frac{-2}{\sqrt{5}}$

$x = -\sqrt{5}$

$\cos\alpha = \frac{-\sqrt{5}}{3}$

$\tan\alpha = \frac{-2}{-\sqrt{5}} = \frac{2}{\sqrt{5}}$

Ex 5. The tangent of a given angle α is equal to 5. Find $\sin\alpha$ and $\cos\alpha$ given that the terminal arm of the angle α is in the third quadrant.	$\tan\alpha = 5 = \frac{y}{x} = \frac{-5}{-1}$; $x=-1$, $y=-5$, $R=\sqrt{1+25}=\sqrt{26}$; $\alpha \in Q\ III$ (third) ; $\sin\alpha = \frac{y}{R} = \frac{-5}{\sqrt{26}}$; $\cos\alpha = \frac{x}{R} = \frac{-1}{\sqrt{26}}$; $P(-1,-5)$

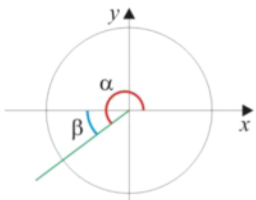
H First Quadrant

Ex 6. The exact values of the functions sine, cosine, and tangent for some angles in the first quadrant are:

α	$0 = 0°$	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\sin\alpha$	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2} = 1$
$\cos\alpha$	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{0}}{2} = 0$
$\tan\alpha$	$\frac{0}{1} = 0$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\frac{1}{0} = $ und

I Related Angle

The related angle β is the angle between the terminal arm of an angle α and the x-axis.



The following relations are true:

$\sin\alpha = \pm\sin\beta$
$\cos\alpha = \pm\cos\beta$
$\tan\alpha = \pm\tan\beta$

Ex 7. Use the related angle property to find the exact value of the trigonometric functions for each angle.

a) $\sin\frac{2\pi}{3}$; $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$; $\sin\frac{2\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

b) $\cos\frac{5\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

c) $\tan\frac{7\pi}{4} = -\tan\frac{\pi}{4} = -1$

J Co-terminal Angles

Co-terminal angles have the same value for the trigonometric functions.
To find the value of the trigonometric functions of a given angle, find first a co-terminal angle in the interval $[0,2\pi]$ and then use the related angle.

→ same terminal arm

Ex 8. Find the exact value for each angle.

a) $\sin\frac{11\pi}{3} = \sin\left(2\pi + \frac{5\pi}{3}\right) = \sin\frac{5\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$ (coterminal; related)

b) $\cos\frac{17\pi}{6} = \cos\left(2\pi + \frac{5\pi}{6}\right) = \cos\frac{5\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$

c) $\tan\frac{21\pi}{4} = \tan\left(4\pi + \frac{5\pi}{4}\right) = \tan\frac{5\pi}{4} = \tan\frac{\pi}{4} = 1$

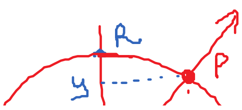
Reading: Nelson Textbook, Pages 323-329
Homework: Nelson Textbook, Page 330: #5, 6, 7, 8, 13, 18, 20

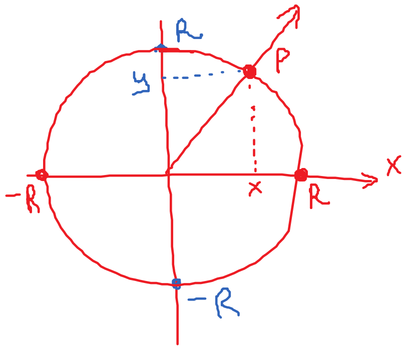
Sine and Cosine

$D = \mathbb{R} = (-\infty, \infty)$ (any real number)

$-R \le x \le R$

$-1 \le \frac{x}{R} \le 1$





$-R \le x \le R$

$-1 \le \frac{x}{R} \le 1$

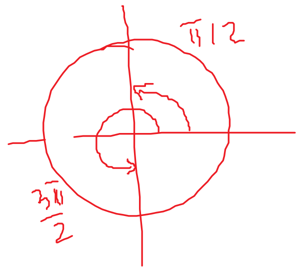
$\boxed{-1 \le \cos\alpha \le 1}$

$\boxed{R_{\cos} = [-1, 1]} = R_{\sin}$

$-R \le y \le R$

$-1 \le \frac{y}{R} = \sin\alpha \le 1$

Tangent Function

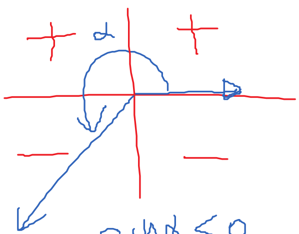


$D_{\tan} = \{x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + k\pi\}$, $k \to 0, \pm 1, \pm 2, \ldots$

$R_{\tan} = \mathbb{R}$

Sign of the Sine

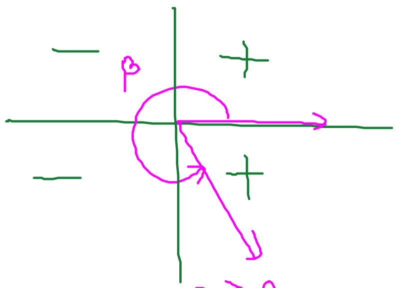
$\sin\alpha = \frac{y}{R}$



$\sin\alpha < 0$

Sign of the Cosine

$\cos\alpha = \frac{x}{R}$

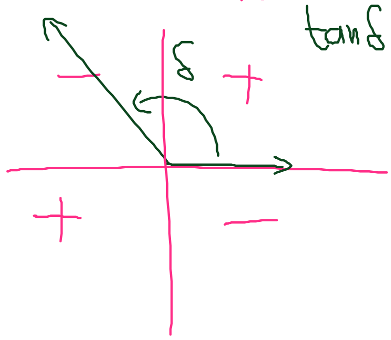


$\cos\beta > 0$

Sign of Tangent

$\tan\alpha = \frac{y}{x}$

$\tan\delta < 0$



CAST Rule

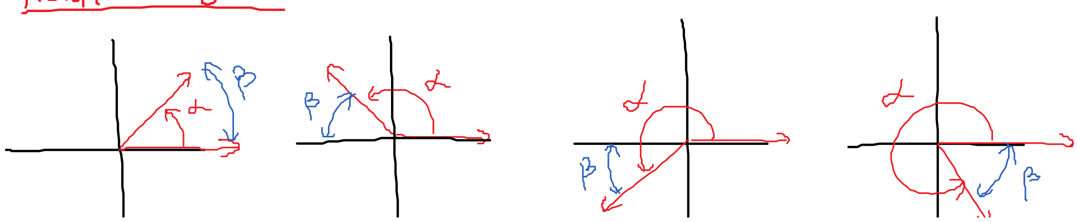
Only Sine is + here → S

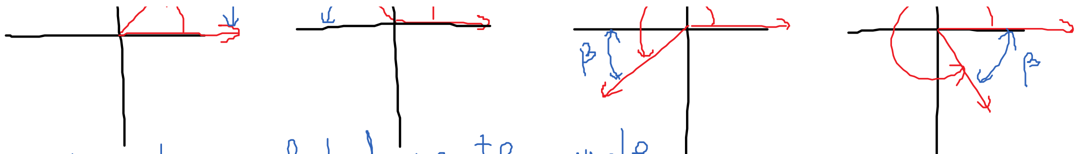
A ← All are positive here

Only Tangent is + here → T

C ← Only Cosine is + here

Related angle

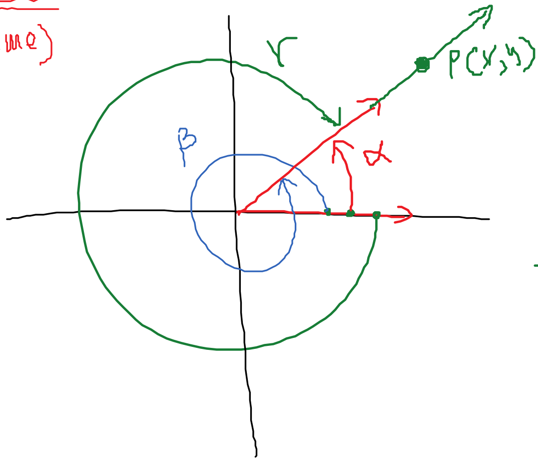




β is the related acute angle

Coterminal Angles

(same)



$\sin\alpha = \sin\beta = \sin\gamma$

$\cos\alpha = \cos\beta = \cos\gamma$

$\tan\alpha = \tan\beta = \tan\gamma$

Note If α and β are coterminal then

$$\boxed{\alpha - \beta = k(2\pi) = k(360^\circ)}$$

$k = 0, \pm 1, \pm 2, \dots$