5.2 Exploring Quotients of Polynomial Functions (Rational Functions)

A Rational Functions
A rational function is a function of the form:

\[ f(x) = \frac{P(x)}{Q(x)} \]

where \( P(x) \) and \( Q(x) \) are polynomial functions.

Ex 1. Verify if the following functions are or are not rational functions.

\( a) \ f(x) = \frac{x^2 - x + 2}{x^2 - 1} \)
\( b) \ f(x) = \frac{x}{\sqrt{x} + x^2} \)
\( c) \ f(x) = \frac{x^2 + 1}{x} \)

B Domain
The domain of a rational function is determined by the restriction

\( Q(x) \neq 0 \)

Ex 2. Find the domain of each rational function.

\( a) \ f(x) = \frac{x^2 - 1}{x - 1} \)
\( b) \ f(x) = \frac{x + 1}{x^2 - 4} \)
\( c) \ f(x) = \frac{x^2 - x}{6x^3 + x^2 - 2x} \)

C y-intercept Point
The y-intercept point for any function \( y = f(x) \) is the point \((0, f(0))\) if \( 0 \) is in the domain of the function \( f \).

Ex 3. Find the y-intercept of each rational function.

\( a) \ f(x) = \frac{x^2 + 1}{x} \)
\( b) \ f(x) = \frac{x + 3}{x^2 - 1} \)

D Holes
The rational function \( f(x) = \frac{P(x)}{Q(x)} \) has a hole in its graph at \( x = a \) if

\( P(a) = 0 \) and \( Q(a) = 0 \)

and if the simplified formula of function \( f(x) \) is defined at \( x = a \).

Ex 4. Sketch the graph of the following rational functions.

\( a) \ f(x) = \frac{x^2}{x} \)
\( b) \ f(x) = \frac{x^2}{x^4} \)
\( c) \ f(x) = \frac{x^2 - 4}{x + 2} \)
\( d) \ f(x) = \frac{x^3 - 1}{x - 1} \)

E Zeros
The rational function \( f(x) = \frac{P(x)}{Q(x)} \) has a zero at \( x = a \) if

\( P(a) = 0 \) and \( Q(a) \neq 0 \)

Ex 5. Find the zeros of the following rational functions.

\( a) \ f(x) = \frac{x - 1}{x^2 - 1} \)
\( b) \ f(x) = \frac{x^2 - 1}{x + 1} \)
\( c) \ f(x) = \frac{x^2 + 1}{x - 1} \)
### F Vertical Asymptotes
The vertical line \( x = a \) is a **vertical asymptote** for the graph of the function \( f(x) \) if the value of the function becomes **unbounded** (\( y = f(x) \to \pm \infty \)) as \( x \) approaches \( a \) from the left or from the right.

Note. If \( x = a \) is a vertical asymptote for a rational function \( f(x) = \frac{P(x)}{Q(x)} \) then (after simplification)
\[
P(a) \neq 0 \text{ and } Q(a) = 0
\]

Note. A vertical asymptote **splits** the graph of a function in branches.

### G Horizontal Asymptotes
The horizontal line \( y = c \) is a **horizontal asymptote** for the graph of the function \( f(x) \) if \( y = f(x) \to a \) as \( |x| \) becomes **unbounded** (\( x \to \pm \infty \)).

Note. Some functions may have two different horizontal asymptotes (one as \( x \to \infty \) and one as \( x \to -\infty \)).

Note. Rational functions may have **at most one** horizontal asymptote.

In the case of a rational function:
\[
f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}
\]

- If \( P(x) \) and \( Q(x) \) have the **same degree** (\( n = m \))
  then the equation of the horizontal asymptote is \( y = \frac{a_n}{b_m} \).

- If the degree of \( P(x) \) is less that the degree of \( Q(x) \) then the equation of the horizontal asymptote is \( y = 0 \).

- If the degree of \( P(x) \) is greater that the degree of \( Q(x) \) then the rational function **does not have** a horizontal asymptote.

### H Graph Sketching
Use the x-intercepts, y-intercept, symmetry, vertical and horizontal asymptote to sketch the graph of a rational function.

### Ex 6. For each case, find the equation of the vertical asymptotes.
\[
a) \quad f(x) = \frac{x}{x^2 + 1}
\]
\[
b) \quad f(x) = \frac{x + 3}{x^2 - x - 6}
\]
\[
c) \quad f(x) = \frac{x^3 - 1}{x^2 - 1}
\]

### Ex 7. For each case, find the equation of the horizontal asymptote (if exists).
\[
a) \quad f(x) = \frac{x^2 - 2x + 1}{3x^3 - 3x + 5}
\]
\[
b) \quad f(x) = \frac{2x^2 + 3}{x + 1}
\]
\[
c) \quad f(x) = -\frac{x^3 + x - 1}{(2x - 1)(x + 1)^2}
\]

### Ex 8. Sketch the graph for the following rational functions.
\[
a) \quad f(x) = \frac{x - 1}{x^2 - 1}
\]
\[
b) \quad f(x) = \frac{x^2 - 1}{x^2 - 4}
\]

**Reading:** Nelson Textbook, Pages 258-261

**Homework:** Nelson Textbook, Page 262: #1, 2, 3