4.1 Solving Polynomial Equations

A Polynomial Equation

A *polynomial equation* is defined as:

$$P(x) = 0$$

where $P(x)$ is a polynomial function.

Note. The numbers x satisfying the polynomial equation are called the *roots* or the *solutions* of the polynomial equation.

Note. The *roots (solutions)* of the polynomial equation $P(x) = 0$ are the same as the *zeros* of the polynomial function $y = P(x)$.

Ex 1. Show that $x = \sqrt{3}$ is a solution of the polynomial equation

$x^4 + 9 = 6x^2$

$LS = (\sqrt{3})^4 + 9 = 18$

$RS = 6(\sqrt{3})^2 = 18$

$LS = RS$

LS ⇒ Left Side

RS ⇒ Right Side

B Grouping

~~Any~~ Some polynomial equation may be solved by grouping terms adequately.

Note. Is not easy to see how to group terms in order to solve the equation.

Find the solutions (roots)

Ex 2. Solve for x by grouping.

$8x^3 - 12x^2 - 2x + 3 = 0$

$4x^2(2x-3) - (2x-3) = 0$

$(2x-3)(4x^2-1) = 0$

$2x-3$ or $4x^2-1=0$

$\therefore x = \frac{3}{2}$ or $x = \pm\frac{1}{2}$

$x^2 = \frac{1}{4}$

C Integral Zeros Theorem

If $x = b$ is an *integral zero* of the polynomial $P(x)$ with integral coefficients, then b is a *factor (divisor)* of the *constant term* a_0 of the polynomial.

Note. A *real zeros* of the polynomial function $P(x)$ is also called *x-intercept* because the graph *touches* or *crosses* the x-axis at this number.

$(x+1)(x^2+2x-15) = 0$

$(x+1)(x+5)(x-3) = 0$

$\therefore x = -1, -5, 3$

Ex 3. Solve for x by looking first at integral roots (solutions).

$(x^2 - 13)x = 15 - 3x^2$

$P(x) = x^3 + 3x^2 - 13x - 15 = 0$

$\{\pm 1, \pm 3, \pm 5, \pm 15\}$

$P(-1) = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -13 & -15 \\ & 0 & -1 & -2 & 15 \\ \hline & 1 & 2 & -15 & 0 \end{array}$$

$x^2 + 2x - 15$

D Rational Zero Theorem

If $x=\frac{b}{a}$ is an *rational zero* of the polynomial $P(x)$ with integral coefficients, then b is a *factor (divisor)* of the *constant term* a_0 and a is a *factor (divisor)* of the *leading coefficient* a_n.

Ex 4. Solve for x by looking first at rational roots (solutions).

$$3x^4+\frac{7}{2}x^3-\frac{2}{3}x^2-\frac{3}{2}x-\frac{1}{3}=0$$

Ex 5. Is $-2/3$ a possible rational zero of $P(x)=2x^3+...+-8$? Explain. Not possible
3 is not a factor of 2

E Non Rational Zeros

If $x=a+b\sqrt{c}$ (a,b,c are rational numbers, $c>0$) is a zero of a polynomial with integral coefficients $P(x)$, then $x=a-b\sqrt{c}$ is also a zero of this polynomial.

$$\begin{array}{r|l} & x^2-3 \\ \hline x^2-2x-1 & x^4-2x^3-4x^2+6x+3 \\ & \cancel{x^4}-\cancel{2x^3}-x^2 \\ \hline & -3x^2+6x+3 \\ & -3x^2+6x+3 \\ \hline & 0 \end{array}$$

$\therefore x=1\pm\sqrt{2},\ \pm\sqrt{3}$

Ex 6. Solve for x

$$\underbrace{x^4-2x^3-4x^2+6x+3}_{P(x)}=0$$

given that $x=1+\sqrt{2}$ is one of its roots.

$\{\pm1,\pm3\}$ no integral zeros

no rational zeros

$x_1=1+\sqrt{2}$

$x_2=1-\sqrt{2}$

$(x-x_1)(x-x_2)=(x-1-\sqrt{2})(x-1+\sqrt{2})$

$=(x-1)^2-(\sqrt{2})^2=x^2-2x-1$ is a factor of $P(x)$

F Technology

In some cases the real roots of a polynomial equation may be found only using *technology*.

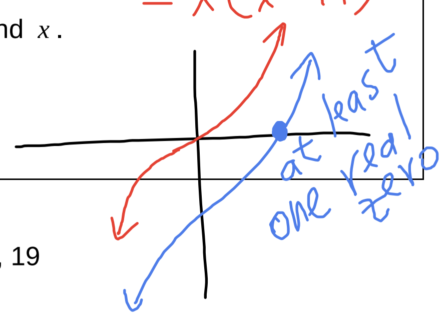
Ex 7. How many real roots does the equation given below have?

$$x^3+x-1=0$$

Use technology to find x.

$f(x)=x^3+x$

$=x(x^2+1)$



Reading: Nelson Textbook, Pages 196-204
Homework: Nelson Textbook, Page 204: #6ad, 7bf, 8a, 9b, 10, 13, 15, 16, 18, 19

④ $\underbrace{3x^4+\frac{7}{2}x^3-\frac{2}{3}x^2-\frac{3}{2}x-\frac{1}{3}}_{P(x)}=0 \quad \Big| \cdot 6$

$F(x)=6P(x)=18x^4+21x^3-4x^2-9x-2=0$

$\left\{\frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18}\right\}$

$F(-1)=0 \qquad F(-\frac{1}{2})=0$

$(x+1)(x+\frac{1}{2})(18x^2-6x-4)=0$

$$\begin{array}{r|rrrrr}
-1 & 18 & 21 & -4 & -9 & -2 \\
 & 0 & -18 & -3 & 7 & 2 \\
\hline
-\frac{1}{2} & 18 & 3 & -7 & -2 & 0 \\
 & 0 & -9 & 3 & 2 & \\
\hline
 & 18 & -6 & -4 & 0 &
\end{array}$$

$\therefore x=-1, -\frac{1}{2}, \frac{2}{3}, -\frac{1}{3}$

$9x^2-3x-2=0$

$x=\frac{3\pm\sqrt{9+72}}{18}$

$=\frac{3\pm 9}{18} \to \frac{12}{18}=\frac{2}{3}, \quad -\frac{6}{18}=-\frac{1}{3}$