4.1 Solving Polynomial Equations

A Polynomial Equation

A *polynomial equation* is defined as:

$$\boxed{P(x)=0}$$

where $P(x)$ is a polynomial function.

may be real or complex

Note. The numbers x satisfying the polynomial equation are called the *roots* or the *solutions* of the polynomial equation.

Note. The *roots (solutions)* of the polynomial equation $P(x)=0$ are the same as the *zeros* of the polynomial function $y=P(x)$.

Ex 1.

Show that $x=\sqrt{3}$ is a solution of the polynomial equation

$x^4+9=6x^2 \Rightarrow \underbrace{x^4-6x^2+9}_{P(x)}=0$

$(\sqrt{3})^4-6(\sqrt{3})^2+9$

$=9-6(3)+9$

$=0$ $\therefore \sqrt{3}$ is a solution of the given equation

B Grouping

~~Any~~ Some polynomial equation may be solved by grouping terms adequately.

Note. Is not easy to see how to group terms in order to solve the equation.

As a Solution set

$x\in\left\{\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}\right\}$

Set notation

Ex 2.

Solve for x by grouping.

$8x^3-12x^2-2x+3=0$

$4x^2(2x-3)-(2x-3)=0$

$(2x-3)(4x^2-1)=0$

$2x-3=0$ OR ~~and~~ $4x^2-1=0 \Rightarrow x^2=\frac{1}{4}$

$\therefore x=\frac{3}{2}$ or $x=\pm\frac{1}{2}$

$\therefore x=\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}$ (as a list)

C Integral Zeros Theorem

If $x=b$ is an *integral zero* of the polynomial $P(x)$ with integral coefficients, then b is a *factor (divisor)* of the *constant term* a_0 of the polynomial.

Note. A *real zeros* of the polynomial function $P(x)$ is also called *x-intercept* because the graph *touches* or *crosses* the x-axis at this number.

Ex 3.

Solve for x by looking first at integral roots (solutions).

$(x^2-13)x=15-3x^2 \Rightarrow \underbrace{x^3+3x^2-13x-15}_{P(x)}=0$

List of possible integral zeros: $\pm1, \pm3, \pm5, \pm15$

$P(1)=1+3-13-15\neq0$

$P(-1)=-1+3+13-15=0$ ✓

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -13 & -15 \\ & 0 & -1 & -2 & 15 \\ \hline & 1 & 2 & -15 & 0 \end{array}$$

$(x+1)(x^2+2x-15)=0$

$(x+1)(x+5)(x-3)=0$

$\therefore x=-1, -5, 3$

D Rational Zero Theorem

If $x = \frac{b}{a}$ is an *rational zero* of the polynomial $P(x)$ with integral coefficients, then b is a *factor (divisor)* of the *constant term* a_0 and a is a *factor (divisor)* of the *leading coefficient* a_n.

$P(1) = 18+21-4-9-2 \neq 0$

$P(-1) = 18-21-4+9-2 = 0$ ✓

$P(2) = 18(16)+21(8)-4(4)-9(2)-2 \neq 0$

$P(-2) = 18(16)+21(-8)-4(4)-9(-2)-2 \neq 0$

$P\left(\frac{1}{2}\right) = 18\left(\frac{1}{16}\right)+21\left(\frac{1}{8}\right)-4\left(\frac{1}{4}\right)-9\left(\frac{1}{2}\right)-2 \neq 0$ ✓

$P\left(-\frac{1}{2}\right) = 18\left(\frac{1}{16}\right)-21\left(\frac{1}{8}\right)-4\left(\frac{1}{4}\right)+9\left(\frac{1}{2}\right)-2 = 0$

Ex 4. Solve for x by looking first at rational roots (solutions).

$$3x^4 + \frac{7}{2}x^3 - \frac{2}{3}x^2 - \frac{3}{2}x - \frac{1}{3} = 0 \quad \Big| \cdot 6$$

$18x^4 + 21x^3 - 4x^2 - 9x - 2 = 0$

List of rational zeros :

$$\frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18}$$

Ex 5. Is $-2/3$ a possible rational zero of $P(x) = 2x^3 + \ldots + -8$? Explain.

Not possible

3 is not a factor of 2

E Non Rational Zeros

(optional)

If $x = a + b\sqrt{c}$ (a, b, c are rational numbers, $c > 0$) is a zero of a polynomial with integral coefficients $P(x)$, then $x = a - b\sqrt{c}$ is also a zero of this polynomial.

$a - \sqrt{b}$

$a + \sqrt{b}$

Ex 6. Solve for x $P(x)$

$x_2 = 1 - \sqrt{2}$

$x^4 - 2x^3 - 4x^2 + 6x + 3 = 0 \Rightarrow (x-x_1)(x-x_2)(x^2-3) = 0$

given that $x_1 = 1 + \sqrt{2}$ is one of its roots.

$d(x) = (x-x_1)(x-x_2) = (x-1-\sqrt{2})(x-1+\sqrt{2}) = (x-1)^2 - (\sqrt{2})^2$

$= x^2 - 2x + 1 - 2$

$= x^2 - 2x - 1$

$$\begin{array}{r|l} & x^2 - 3 \\ x^2 - 2x - 1 & x^4 - 2x^3 - 4x^2 + 6x + 3 \\ & x^4 - 2x^3 - x^2 \\ \hline & -3x^2 + 6x + 3 \\ & -3x^2 + 6x + 3 \\ \hline & 0 \end{array}$$

$\therefore x = 1 \pm \sqrt{2}, \pm\sqrt{3}$

F Technology

In some cases the real roots of a polynomial equation may be found only using *technology*.

List : ± 1

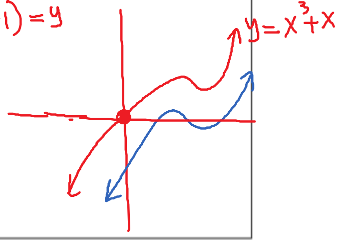
Ex 7. How many real roots does the equation given below have? At least one

$x^3 + x - 1 = 0$

$x(x^2+1) = y$

Use technology to find x.

$\therefore x \simeq 0.6823$



Reading: Nelson Textbook, Pages 196-204
Homework: Nelson Textbook, Page 204: #6ad, 7bf, 8a, 9b, 10, 13, 15, 16, 18, 19

Academic Support

Mon and Wed

8 AM → 8.30 AM

11 45 AM → 12.15

-1	18	21	-4	-9	-2
	0	-18	-3	7	2
$-1/2$	18	3	-7	-2	0
	0	-9	3	2	
	18	-6	-4	0	

$$(x+1)\left(x+\frac{1}{2}\right)\left(18x^2-6x-4\right)=0$$

$$18x^2-6x-4=0$$

$$9x^2-3x-2=0$$

$$x=\frac{3\pm\sqrt{9-4(9)(-2)}}{2(9)}$$

$$=\frac{3\pm 9}{18}$$

$$\frac{12}{18}=\frac{2}{3}$$

$$-\frac{6}{18}=-\frac{1}{3}$$

$$\therefore\ x=-1,\ -\frac{1}{2},\ \frac{2}{3},\ -\frac{1}{3}$$