3.6 Factoring Polynomials

A The Remainder Theorem

If a polynomial $P(x)$ is divided by $x-b$ then the remainder is $r = P(b)$.

Proof: $P(x) = (x-b)Q(x) + r(x)$

$P(b) = \cancel{(b-b)Q(b)} + r(b)$

$\boxed{P(b) = r}$

Ex 1. Determine the remainder when $P(x) = 2x^3 - 4x^2 + 3x - 6$ is divided by

a) $x-2$ $\Rightarrow r = P(2)$

$= 2(2^3) - 4(2^2) + 3(2) - 6 = 0$

b) $x+1$

$r = P(-1) = -2 - 4 - 3 - 6 = -15$

Ex 2. When $P(x) = x^3 - kx^2 + 17x + 6$ is divided by $x-3$, the remainder is 12. Find the value of k.

$12 = P(3)$

$12 = 27 - 9k + 51 + 6$

$9k = 72$

$\therefore \boxed{k = 8}$

therefore

Ex 3. When a polynomial $P(x) = 3x^3 + cx^2 + dx - 7$ is divided by $x-2$, the remainder is -3. When $P(x)$ is divided by $x+1$, the remainder is -18. What are the values of c and d?

$-3 = P(2) \Rightarrow -3 = 24 + 4c + 2d - 7$ (1)

$-18 = P(-1) \Rightarrow -18 = -3 + c - d - 7$ (2)

$4c + \cancel{2d} = -20$ (1)

$c - d = -8$ (2) $\times 2$

$2c - \cancel{2d} = -16$ (2)

(1) + (2)

$6c = -36$

$\therefore \boxed{c = -6}$

$\rightarrow -6 - d = -8$

$\therefore \boxed{d = 2}$

B The Remainder Theorem (II)

If a polynomial $P(x)$ is divided by $ax-b$ then the remainder is $r = P(b/a)$.

$ax - b = 0$

$x = \frac{b}{a}$

Proof:

$P(x) = (ax-b)Q(x) + r(x)$

$P\left(\frac{b}{a}\right) = \cancel{\left(a \cdot \frac{b}{a} - b\right) Q\left(\frac{b}{a}\right)} + r$

Ex 4. Determine the remainder when $P(x) = 2x^3 + 3x^2 - 7x - 3$ is divided by $2x+5$.

$r = P\left(-\frac{5}{2}\right)$

$= 2\left(-\frac{5}{2}\right)^3 + 3\left(-\frac{5}{2}\right)^2 - 7\left(-\frac{5}{2}\right) - 3$

$\therefore r = 2$

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C The Factor Theorem

A polynomial $P(x)$ has $x-b$ as a *factor* if and only if $P(b)=0$.

Note. In this case b is a *zero* of the polynomial function $P(x)$.

$P(x) = (x-b)Q + r$

$r \leftrightarrow P(b) = 0$

Ex 5. Determine whether

a) $x+2$ is a factor of $P(x)=x^3+5x^2+2x-8$

$r = P(-2) = -8+20-4-8 = 0$

$\therefore$ Yes, $x+2$ is a factor of $P(x)$

b) x^2-1 is a factor of $P(x)=2x^4-3x^3-x^2+3x-1$

$x^2-1 = (x-1)(x+1)$

$P(1)=0$ and $P(-1)=0$

$\therefore$ x^2-1 is a factor

D Integral Zero Theorem

is an integer

If $x=b$ is an *integral zero* of the polynomial $P(x)$ with *integral coefficients*, then b is a *factor* (divisor) of the *constant term* a_0 of the polynomial.

$b \in \{\text{factors of } a_0\}$

c) $P(x)=x^4-2x^3-x^2+4x-2$

Ex 6. Factor completely $\Rightarrow a(x-x_1)(x-x_2)\cdots$

a) $P(x)=x^4-x^3-7x^2+x+6$

Any integral zero $\in \{\pm1, \pm2, \pm3, \pm6\}$

$P(1)=0$

$P(-1)=0$

$P(2)=-12$

$P(-2)=0$

$$\begin{array}{r|rrrrr}
1 & 1 & -1 & -7 & 1 & 6 \\
 & 0 & 1 & 0 & -7 & -6 \\
\hline
-1 & 1 & 0 & -7 & -6 & 0 \\
 & 0 & -1 & 1 & 6 & \\
\hline
-2 & 1 & -1 & -6 & 0 & \\
 & 0 & -2 & 6 & & \\
\hline
 & 1 & -3 & 0 & &
\end{array}$$

~~b) $P(x)=2x^3+3x^2-3x-2$~~

$\therefore P(x) = (x-1)(x+1)(x+2)(x-3)$

E Rational Zero Theorem

If $x=b/a$ is an *rational zero* of the polynomial $P(x)$ with *integral coefficients*, then b is a *factor* (divisor) of the *constant term* a_0 and a is a *factor* (divisor) of the *leading term* a_n.

Ex 7. Factor completely.

$P(x)=12x^4-4x^3-11x^2+x+2$

Reading: Nelson Textbook, Pages 171-176
Homework: Nelson Textbook, Page 176: #1, 2, 5, 6ab, 7af, 9, 10, 12, 13, 16

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Integral Zero Theorem (Proof)

$P(x) = a_n x^n + \cdots + a_1 x + a_0$ integers

If b is a zero

$P(b) = a_n b^n + \cdots + a_1 b + a_0 = 0$

$a_0 = -a_n b^n - \cdots - a_1 b$ $\Rightarrow$ b is a factor of a_0

$a_0 = b\,\underbrace{\left(-a_n b^{n-1} \cdots - a_1\right)}_{\text{integer}}$

(6b) $P(x) = 2x^3 + 3x^2 - 3x - 2 = (x-1)(x+2)(2x+1)$

Integral zeros $\in \{\pm 1, \pm 2\}$ $= 2(x-1)(x+2)(x+\frac{1}{2})$

$-\frac{1}{2}$

$P(1) = 0$

$P(-1) = 2$

$P(2) = 20$

$P(-2) = 0$

1	2	3	-3	-2
	0	2	5	2
	2	5	2	0

-2	2	5	2
	0	-4	-2
	2	1	0

(6c) $P(x) = x^4 - 2x^3 - x^2 + 4x - 2 = (x-1)^2(x^2-2)$

$\{\pm 1, \pm 2\}$

$\boxed{P(1) = 0}$ ✓

$P(-1) = -4$ ✗

$P(2) = 2$ ✗

$P(-2) = 18$ ✗

$$\begin{array}{r|rrrrr} 1 & 1 & -2 & -1 & 4 & -2 \\ & 0 & 1 & -1 & -2 & 2 \\ \hline 1 & 1 & -1 & -2 & 2 & 0 \\ & 0 & 1 & 0 & -2 & \\ \hline & 1 & 0 & -2 & 0 & \end{array}$$

$x_1 = 1$, $m_1 = 2$

$x_2 = \sqrt{2}$, $m_2 = 1$

$x_3 = -\sqrt{2}$, $m_3 = 1$

$\therefore P(x) = (x-1)^2(x-\sqrt{2})(x+\sqrt{2})$

(7) $P(x) = 12x^4 - 4x^3 - 11x^2 + x + 2 = (x-1)(x-\frac{1}{2})(12x^2+14x+4)$

Any rational zero $\in \left\{ \dfrac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12} \right\}$

$P(1) = 0$ ✓

$P(-1) = 6$

$P(2) = 120$

$P(-2) = 180$

$P(\frac{1}{2}) = 0$ ✓

$P(-\frac{1}{2}) =$

$$\begin{array}{r|rrrrr} 1 & 12 & -4 & -11 & 1 & 2 \\ & 0 & 12 & 8 & -3 & -2 \\ \hline 1/2 & 12 & 8 & -3 & -2 & 0 \\ & 0 & 6 & 7 & 2 & \\ \hline & 12 & 14 & 4 & 0 & \end{array}$$

$\therefore P(x) = 12(x-1)(x-\frac{1}{2})(x+\frac{2}{3})(x+\frac{1}{2})$

$12x^2 + 14x + 4 = 0$

$6x^2 + 7x + 2 = 0$

$x = \dfrac{-7 \pm \sqrt{49-48}}{12}$ → $\dfrac{-8}{12} = -\dfrac{2}{3}$, $-\dfrac{6}{12} = -\dfrac{1}{2}$