3.5 Dividing Polynomials

A Division of Natural Numbers

If D and $d \neq 0$ are two natural numbers, then there are *unique* numbers q and r such that the following relation (called *division statement*) is true:

$$\frac{D}{d} = q + \frac{r}{d}$$

or: $D = dq + r$

where: $0 \le r < d$

- D is called the *dividend*
- d is called the *divisor*
- q is called the *quotient*
- r is called the *remainder*

Note. If the remainder r is 0 then:

- D is *divisible* by d and
- d is a *factor* of D

Use the division algorithm to get the quotient and the remainder.

Ex 1. Use the division algorithm to find the quotient and the remainder for each division.

a) $\frac{57}{8}$

b) $\frac{23}{-5}$

B Division of Polynomials

If $D(x)$ and $d(x) \neq 0$ are two polynomial functions, then there are two *unique* polynomials $q(x)$ and $r(x)$ such that the following relation (called *division statement*) is true:

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

or: $D(x) = d(x)q(x) + r(x)$

where: $0 \le degree(r) < degree(d)$

- D is called the *dividend*
- d is called the *divisor*
- q is called the *quotient*
- r is called the *remainder*

Note. If the remainder $r(x)$ is 0 then:

- $D(x)$ is *divisible* by $d(x)$ and
- $d(x)$ is a *factor* of $D(x)$

Use the *long division* algorithm to get the quotient and the remainder of the division of two polynomials.

Ex 2. Use the long division algorithm to find the quotient and the remainder for each division.

a) $\frac{6x^3 - 4x^2 - 9x - 3}{2x^2 - 3}$

b) $\frac{x^3 + 1}{x + 1}$

Ex 3. Find a polynomial function $P(x)$ such that, by dividing it to $x^2 - 1$, you get the quotient $2x + 1$ and the remainder $2x - 1$.

Ex 4. What is the polynomial function you have to divide $3x^3 - x^2 - 2x + 6$ to, to get a quotient $x^2 + 1$ and a remainder $-5x + 7$.

C Synthetic Division Algorithm

Synthetic division is a *shorthand* method for dividing a polynomial $P(x)$ by a *linear divisor* $x-b$.

$$
\begin{array}{c|ccccccc}
b & a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_1 & a_0 \\
 & & ba_n & b(a_{n-1}+ba_n) & \dots & & & \\
 & a_n & a_{n-1}+ba_n & a_{n-2}+b(a_{n-1}+ba_n) & \dots & q_1 & q_0 & r
\end{array}
$$

Ex 5. Use the synthetic division algorithm to find the quotient and the remainder.

a) $(-2x^3+3x^2-4x+5)\div(x-2)$

b) $(x^5+2x^3-3)\div(x+3)$

Ex 6. Use the synthetic division to divide:

$$\frac{2x^3-3x^2+5x-7}{2x-1}$$

Ex 7. Use the synthetic division to divide:

$$\frac{x^5-2x^3+2x^2+x-2}{x^2-1}$$

Reading: Nelson Textbook, Pages 162-168
Homework: Nelson Textbook, Page 168: #4, 5bdf, 6bce, 8d, 9c, 10e, 12, 18, 19

(2a) order

$$
\begin{array}{r|l}
 & 3x - 2 \\
\hline
2x^2 - 3 & \cancel{6x^3} - 4x^2 - \cancel{9x} - 3 \\
 & \cancel{6x^3} \qquad\quad\; -\cancel{9x} \quad (-) \\
\hline
 & \cancel{-4x^2} - 3 \\
 & \cancel{-4x^2} + 6 \quad (-) \\
\hline
 & -9
\end{array}
$$

order

$\therefore q(x) = 3x - 2$

$r(x) = -9$

$$\boxed{\frac{6x^3 - 4x^2 - 9x - 3}{2x^2 - 3} = 3x - 2 + \frac{-9}{2x^2 - 3}} \Leftrightarrow \boxed{6x^3 - 4x^2 - 9x - 3 = (3x-2)(2x^2-3) - 9}$$

division statements

(2b)

$$
\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x + 1 & \cancel{x^3} + 1 \\
 & \cancel{x^3} + x^2 \quad (-) \\
\hline
 & \cancel{-x^2} + 1 \\
 & \cancel{-x^2} - x \quad (-) \\
\hline
 & x + 1 \\
 & x + 1 \quad (-) \\
\hline
 & 0
\end{array}
$$

$x^3 + 1 = (x+1)(x^2 - x + 1)$

$\therefore r(x) = 0$

$q(x) = x^2 - x + 1$

$$\frac{x^3 + 1}{x + 1} = x^2 - x + 1$$

$x^3 + 1$ is divisible by $x + 1$

$x + 1$ is a factor of $x^3 + 1$

③ $P(x) = (x^2-1)(2x+1)+2x-1$

④ $3x^3-x^2-2x+6 = \underset{?}{d(x)}(x^2+1)+\overset{\curvearrowleft}{(-5x+7)}$

$$d(x) = \frac{3x^3-x^2-2x+6-(-5x+7)}{x^2+1}$$

$\therefore d(x) = 3x-1$

$$
\begin{array}{r|l}
 & 3x - 1 \\
\hline
x^2+1 & \cancel{3x^3} - x^2 + \cancel{3x} - 1 \\
 & \cancel{3x^3} \quad\quad\; + \cancel{3x} \\
\hline
 & -x^2-1 \\
 & -x^2-1 \\
\hline
 & 0
\end{array}
$$

⑤a $\underbrace{(-2x^3+3x^2-4x+5)}_{\text{must be ordered}} \div \underset{\text{linear}}{(x-2)}$

↳ is the zero of $x-2$

2	-2	3	-4	5
	0	-4	-2	-12
	-2	-1	-6	-7
	$\cdots x^2$	x^1	x^0	r

$\underbrace{\cdots x^2 \quad x^1 \quad x^0}_{q}$

$\therefore r(x) = -7$
(always is a constant number)

$q(x) = -2x^2-x-6$

(5b) $\left(x^5+2x^3-3\right) \div (x+3)$

$$\begin{array}{r|rrrrrr}
 & x^5 & x^4 & x^3 & x^2 & x & x^0 \\
-3 & 1 & 0 & 2 & 0 & 0 & -3 \\
 & 0 & -3 & 9 & -33 & 99 & -297 \\
\hline
 & 1 & -3 & 11 & -33 & 99 & -300 \\
 & x^4 & x^3 & x^2 & x^1 & x^0 & r
\end{array}$$

$$\underbrace{x^4 \quad x^3 \quad x^2 \quad x^1 \quad x^0}_{q}$$

$\therefore r(x) = -300$

$q(x) = x^4 - 3x^3 + 11x^2 - 33x + 99$

(6) $\dfrac{2x^3-3x^2+5x-7}{2x-1} = \dfrac{1}{2}\,\dfrac{2x^3-3x^2+5x-7}{x-\frac{1}{2}} = \dfrac{1}{2}\left(2x^2-2x+4+\dfrac{-5}{x-\frac{1}{2}}\right)$

$$\begin{array}{r|rrrr}
\frac{1}{2} & 2 & -3 & 5 & -7 \\
 & 0 & 1 & -1 & 2 \\
\hline
 & 2 & -2 & 4 & -5
\end{array}$$

$$\underbrace{2 \quad -2 \quad 4}_{q} \quad \underbrace{-5}_{r}$$

$= \underbrace{x^2 - x + 2}_{q(x)} + \dfrac{\overbrace{-5}^{r(x)}}{2x-1}$

⑦ $$\frac{P(x)}{x^2-1} = \frac{P(x)}{(x-1)(x+1)} = \frac{\frac{P(x)}{x-1}}{x+1} = x^3 - x + 2$$

$$\begin{array}{r|rrrrrr}
1 & 1 & 0 & -2 & 2 & 1 & -2 \\
 & 0 & 1 & 1 & -1 & 1 & 2 \\
\hline
-1 & 1 & 1 & -1 & 1 & 2 & 0 \\
 & 0 & -1 & 0 & 1 & -2 & \\
\hline
 & 1 & 0 & -1 & 2 & 0 & \\
 & x^3 & x^2 & x & x^0 & &
\end{array}$$

$\therefore r(x) = 0$

$q(x) = x^3 - x + 2$