3.3 Polynomial Functions in Factored Form

A Simple Zeros

all exponents are 1

Some polynomial functions can be factored in the form:

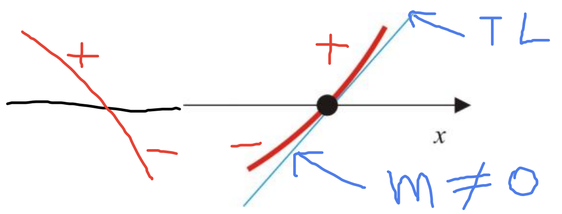
$$f(x)=a_n(x-x_1)(x-x_2)\ldots(x-x_{n-1})(x-x_n)$$

$x_1, x_2, \ldots, x_{n-1}, x_n$ are n *distinct (different) real* numbers and the zeros (or the x-intercepts) of the polynomial function.

$f(x_1)=a_n\underbrace{(x_1-x_1)}_{0}\ldots=0$

Notes:

- The function *changes sign* at each x-intercept.
- The tangent line at each x-intercept *is not horizontal*.

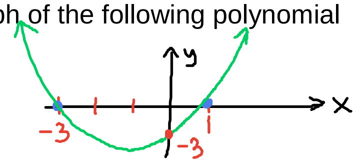


Ex 1. Sketch the graph of the following polynomial functions.

a) $f(x)=(x-1)(x+3)$

$f(0)=-3$

$LC=1>0$



b) $f(x)=(x+1)(x+2)(x+3)$

c) $f(x)=-2(x-1)(x-2)(x+3)$

d) $f(x)=-(x-1)(x+2)(x-3)(x+4)(x-5)$

e) $f(x)=x^3-x^2-2x$

f) $f(x)=(1-x^2)(x^2-4)$

g) $f(x)=x^4-4x^2+3$

B Repeated Zeros

Some polynomial functions can be factored in the form:

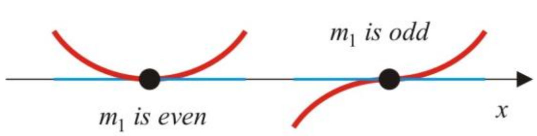
$$f(x)=a_n(x-x_1)^{m_1}(x-x_2)^{m_2}\ldots(x-x_k)^{m_k}$$

x_1 is a zero of *multiplicity (order)* m_1, x_2 is a zero of multiplicity (order) m_2, and so on.

The polynomial function has $m_1+m_2+\ldots+m_k=n$ real zeros (m_1 are coincident (same or identical) and equal to x_1, m_2 are coincident and equal to x_2, and so on).

Notes:

- If m_1 is odd, the function *changes sign* at $x=x_1$ and the graph *crosses* the x-axis.
- If m_1 is even, the function *does not change sign* at $x=x_1$ and the graph *touches* the x-axis.
- If the multiplicity m_1 is greater that 1 then the tangent line at $x=x_1$ is *horizontal*.

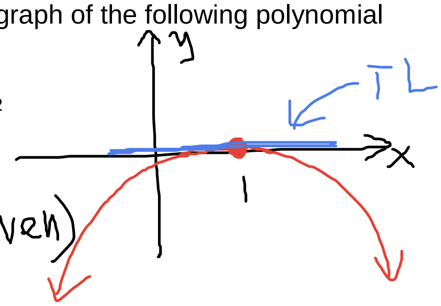


Ex 2. Sketch the graph of the following polynomial functions.

a) $f(x)=-(x-1)^2$

$x_1=1$

$m_1=2$ (even)



b) $f(x)=2(x+1)^3$

c) $f(x)=2(x-1)^2(x+1)^3$

d) $f(x)=-(x+1)(x-2)^2(x+3)^3(x-4)^4$

C Non-real Zeros

A polynomial functions with *non-real zeros* (complex zeros) can be factored as

$$f(x) = (a_1x^2 + b_1x + c)^{m^1} \times \ldots$$

where $\Delta_1 = b_1^{\,2} - 4a_1c_1 < 0, \ldots$

Note. Each trinomial $a_1x^2 + b_1x + c$ has the same sign (the sign of c) for all real numbers x.

Ex 3. Sketch the graph of the following polynomial functions.

a) $f(x) = (x-1)(x+2)^2(x^2+1)$

b) $f(x) = 2(x+1)(x-2)(x-3)^2(x+4)^3(-x^2+x-1)$

Ex 4. Find a polynomial $P(x)$ of degree ~~four~~ six with zeros:

- $x_1 = 1$ of multiplicity $m_1 = 3$
- $x_2 = -2$ of multiplicity $m_2 = 2$
- $x_3 = -1$ of multiplicity $m_3 = 1$

$\therefore P(x) = -\frac{1}{6}(x-1)^3(x+2)^2(x+1)$

such that its graph passes through the point $(2,-8)$.

$y = P(x) = a(x-1)^3(x+2)^2(x+1)$

$(2, -8)$ (x, y)

$-8 = P(2) = a(2-1)^3(2+2)^2(2+1)$

$-8 = a(16)(3) \Rightarrow a = -\frac{1}{6}$

Ex 5. Sketch the graph of the polynomial function: $y = f(x) = (1-x^3)(x^2-4)$.

Ex 6. Sketch the graph of the polynomial function: $y = f(x) = (x^2-4)^2$.

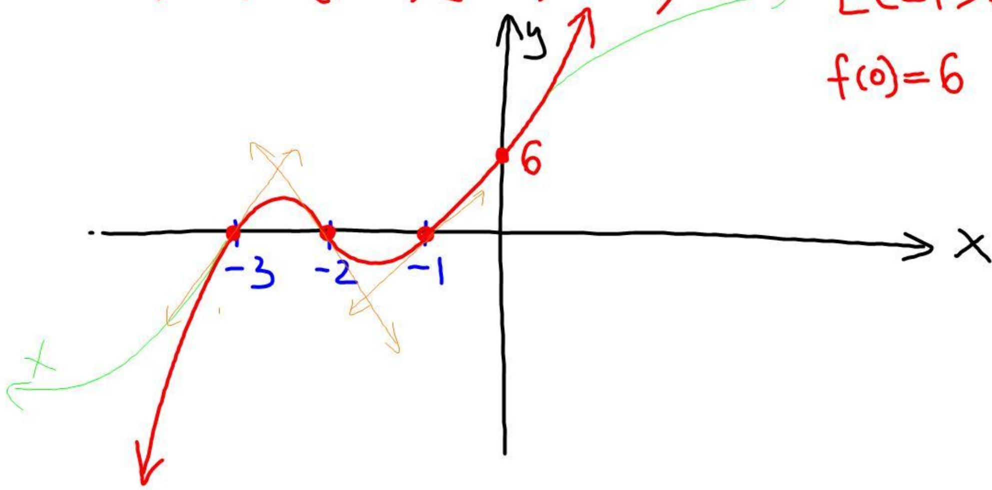
Ex 7. Sketch the graph of the polynomial function: $y = f(x) = |x(x^2-1)|$.

Reading: Nelson Textbook, Pages 139-145
Homework: Nelson Textbook, Page 146: #1, 2, 4, 7, 9ab, 10cd, 13b, 15

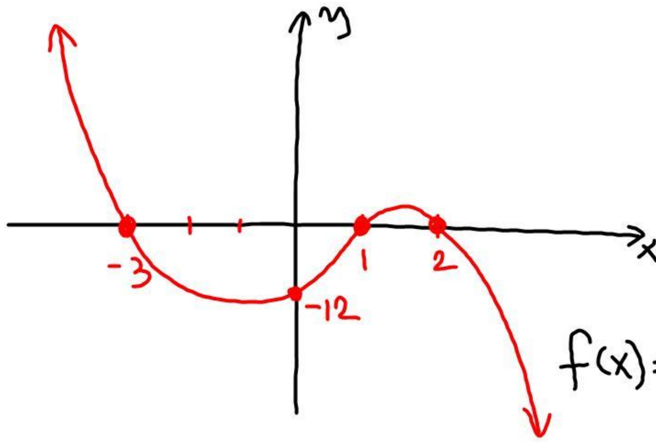
1b) $f(x) = (x+1)(x+2)(x+3)$

$LC = 1 > 0$

$f(0) = 6$



1c)



$f(x) = -2(x-1)(x-2)(x+3)$

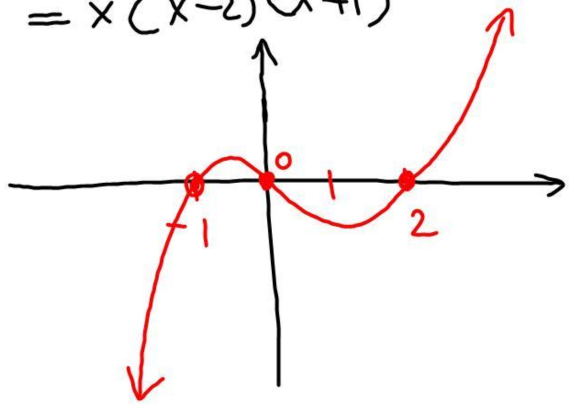
$LC = -2 < 0$

LC=1>0

1e) $f(x) = x^3 - x^2 - 2x$

$= x(x^2 - x - 2)$

$= x(x-2)(x+1)$



1f) $f(x) = (1-x^2)(x^2-4)$

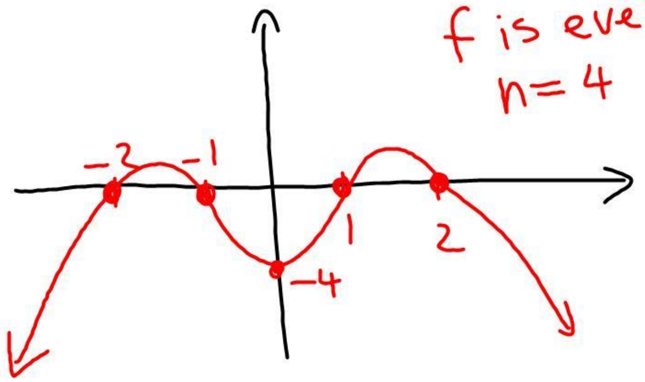
$= (1-x)(1+x)(x-2)(x+2)$

$= -(x-1)(x+1)(x-2)(x+2)$

f is even

n=4 (quartic)

W Shape



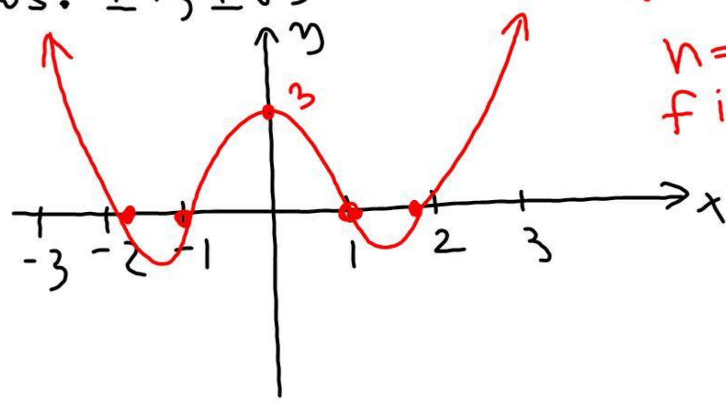
1g) $f(x) = x^4 - 4x^2 + 3$

$LC = 1 > 0$

$= (x^2-1)(x^2-3) = (x-1)(x+1)(x-\sqrt{3})(x+\sqrt{3})$

Zeros: $\pm 1, \pm\sqrt{3}$

$\sqrt{3} \simeq 1.732$

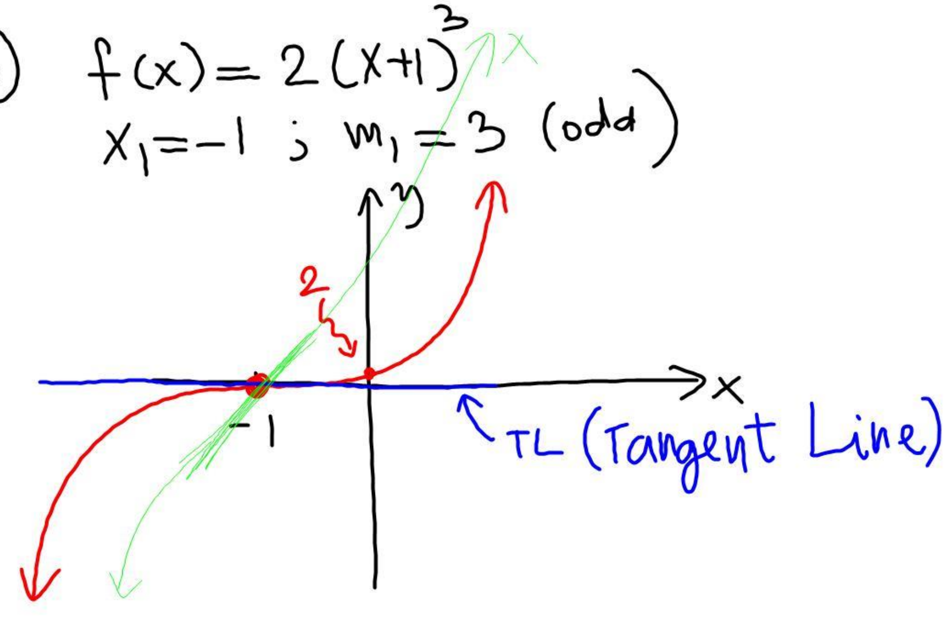


$n = 4$

f is even

2b) $f(x) = 2(x+1)^3$

$x_1 = -1$; $m_1 = 3$ (odd)

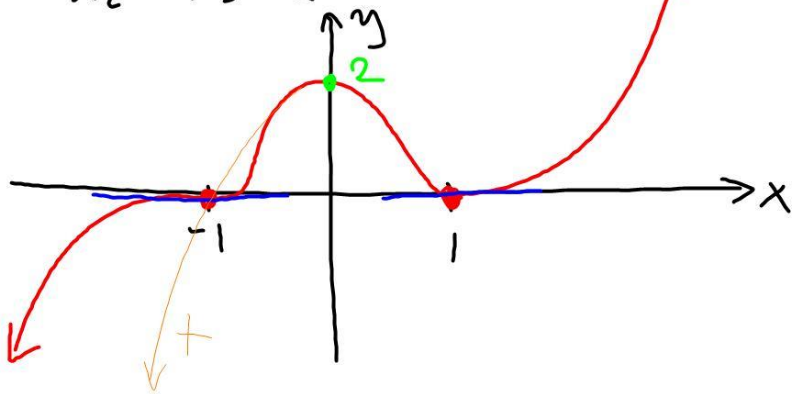


2c) $f(x) = 2(x-1)^2(x+1)^3$

$x_1 = 1\ ;\ m_1 = 2$ (even)

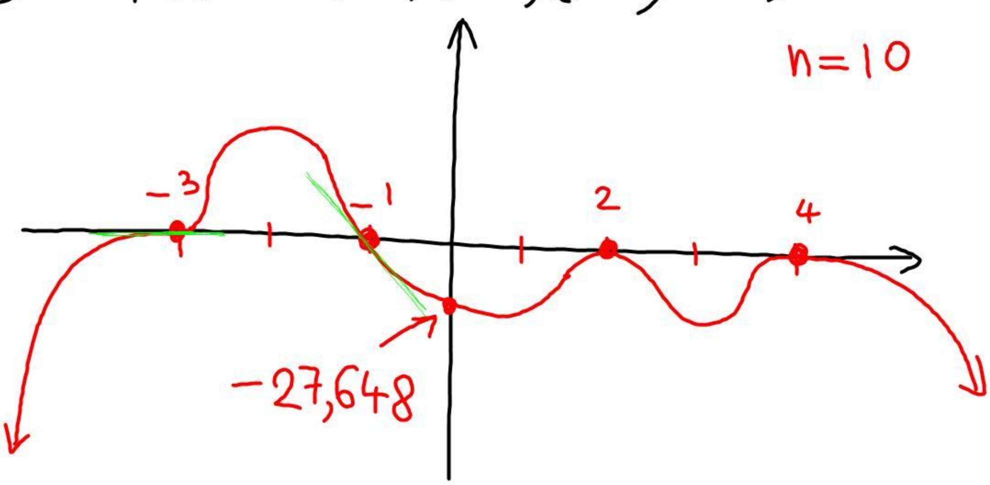
$x_2 = -1\ ;\ m_2 = 3$ (odd)

$n = 2+3$
$= 5$
(quintic)



2d) $f(x) = -(x+1)(x-2)^2(x+3)^3(x-4)^4$

$n = 10$

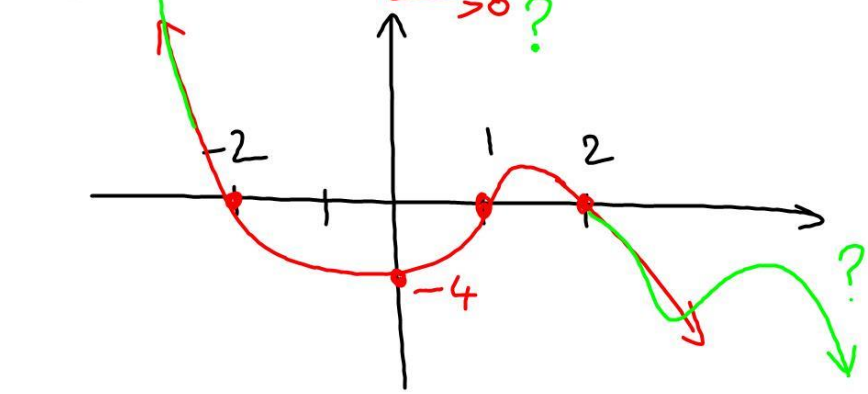


5) $f(x) = (1-x^3)(x^2-4)$

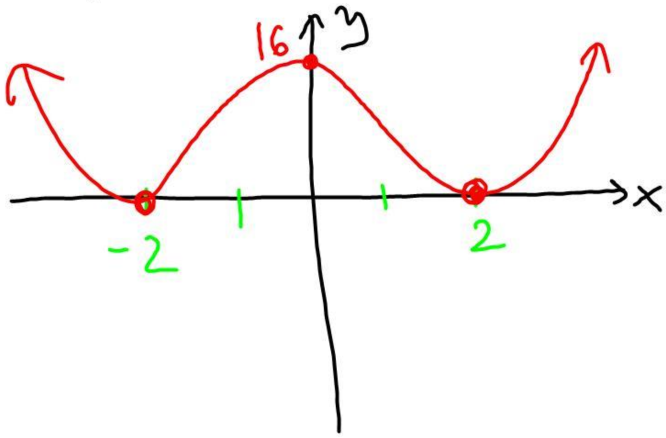
$= -(x^3-1)(x^2-4)$

$= -(x-1)\underbrace{(x^2+x+1)}_{>0\ ?}(x-2)(x+2)$

$D = b^2-4ac$
$= 1^2-4(1)(1)$
$= -3<0$



6) $f(x) = (x^2-4)^2 = [(x-2)(x+2)]^2 = (x-2)^2(x+2)^2$



7) $f(x) = |x(x^2-1)|$

$g(x) = x(x^2-1) = x(x-1)(x+1)$