3.3 Polynomial Functions in Factored Form

A Simple Zeros

Some polynomial functions can be factored in the form:

\[ f(x) = a_n (x-x_1)(x-x_2) \cdots (x-x_{n-1})(x-x_n) \]

\( x_1, x_2, \ldots, x_{n-1}, x_n \) are \( n \) distinct (different) real numbers and the zeros (or the x-intercepts) of the polynomial function.

Notes:
- The function changes sign at each x-intercept.
- The tangent line at each x-intercept is not horizontal.

Ex 1. Sketch the graph of the following polynomial functions.

a) \( f(x) = (x-1)(x+3) \)

b) \( f(x) = (x+1)(x+2)(x+3) \)

c) \( f(x) = -2(x-1)(x-2)(x+3) \)

d) \( f(x) = -(x-1)(x-2)(x-3)(x+4)(x+5) \)

e) \( f(x) = x^3 - x^2 - 2x \)

f) \( f(x) = (1-x^2)(x^2 - 4) \)

g) \( f(x) = x^4 - 4x^2 + 3 \)

B Repeated Zeros

Some polynomial functions can be factored in the form:

\[ f(x) = a_n (x-x_1)^{m_1}(x-x_2)^{m_2} \cdots (x-x_k)^{m_k} \]

\( x_1 \) is a zero of multiplicity (order) \( m_1 \), \( x_2 \) is a zero of multiplicity (order) \( m_2 \), and so on.

The polynomial function has \( m_1 + m_2 + \cdots + m_k = n \) real zeros (\( m_1 \) are coincident (same or identical) and equal to \( x_1 \), \( m_2 \) are coincident and equal to \( x_2 \), and so on).

Notes:
- If \( m_1 \) is odd, the function changes sign at \( x = x_1 \) and the graph crosses the x-axis.
- If \( m_1 \) is even, the function does not change sign at \( x = x_1 \) and the graph touches the x-axis.
- If the multiplicity \( m_1 \) is greater than 1 then the tangent line at \( x = x_1 \) is horizontal.

Ex 2. Sketch the graph of the following polynomial functions.

a) \( f(x) = -(x-1)^2 \)

b) \( f(x) = 2(x+1)^3 \)

c) \( f(x) = 2(x-1)^2(x+1)^3 \)

d) \( f(x) = -(x+1)(x-2)^2(x+3)^3(x-4)^4 \)
### C Non-real Zeros

A polynomial functions with non-real zeros (complex zeros) can be factored as

\[ f(x) = (a_1x^2 + b_1x + c)^{m_1} \times \ldots \]

where \( \Delta_1 = b_1^2 - 4a_1c_1 < 0 \).

Note. Each trinomial \( a_1x^2 + b_1x + c \) has the same sign (the sign of \( c \)) for all real numbers \( x \).

**Ex 3.** Sketch the graph of the following polynomial functions.

a) \( f(x) = (x - 1)(x + 2)^2(x^2 + 1) \)

b) \( f(x) = 2(x + 1)(x - 2)(x - 3)^2(x + 4)^3(-x^2 + x - 1) \)

**Ex 4.** Find a polynomial \( P(x) \) of degree six with zeros:

- \( x_1 = 1 \) of multiplicity \( m_1 = 3 \)
- \( x_2 = -2 \) of multiplicity \( m_2 = 2 \)
- \( x_3 = -1 \) of multiplicity \( m_3 = 1 \)

such that its graph passes through the point \( (2, -8) \).

**Ex 5.** Sketch the graph of the polynomial function:

\[ y = f(x) = (1 - x^3)(x^2 - 4) \]

**Ex 6.** Sketch the graph of the polynomial function:

\[ y = f(x) = (x^2 - 4)^2 \]

**Ex 7.** Sketch the graph of the polynomial function:

\[ y = f(x) = |x(x^2 - 1)| \]

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**Reading:** Nelson Textbook, Pages 139-145

**Homework:** Nelson Textbook, Page 146: #1, 2, 4, 7, 9ab, 10cd, 13b, 15