3.3 Polynomial Functions in Factored Form

A. Simple Zeros

$f(x_1) = a_n(x_1 - x_1)(\cdots) = 0$

Some polynomial functions can be factored in the form:

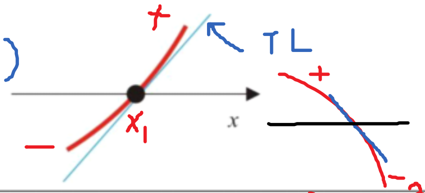
$$f(x) = a_n(x - x_1)(x - x_2)\ldots(x - x_{n-1})(x - x_n)$$

$x_1, x_2, \ldots, x_{n-1}, x_n$ are n *distinct (different) real* numbers and the zeros (or the x-intercepts) of the polynomial function.

Notes:

- The function *changes sign* at each x-intercept.
- The tangent line at each x-intercept *is not horizontal*. TL

(is oblique)
(slant)



Ex 1. Sketch the graph of the following polynomial functions.

a) $f(x) = (x-1)(x+3)$

b) $f(x) = (x+1)(x+2)(x+3)$

c) $f(x) = -2(x-1)(x-2)(x+3)$

d) $f(x) = -(x-1)(x+2)(x-3)(x+4)(x-5)$

e) $f(x) = x^3 - x^2 - 2x$

LC = −1 < 0

f) $f(x) = (1 - x^2)(x^2 - 4)$

$= (1-x)(1+x)(x-2)(x+2)$

g) $f(x) = x^4 - 4x^2 + 3$

$= (x^2-1)(x^2-3)$

$= (x-1)(x+1)(x-\sqrt{3})(x+\sqrt{3})$

B. Repeated Zeros

$(x-x_1)(x-x_1)(x-x_1) = (x-x_1)^3$

Some polynomial functions can be factored in the form:

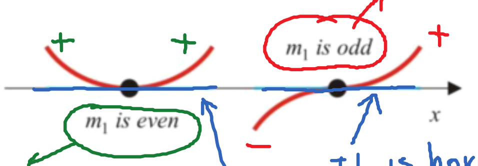
$$f(x) = a_n(x - x_1)^{m_1}(x - x_2)^{m_2}\ldots(x - x_k)^{m_k}$$

x_1 is a zero of *multiplicity* (*order*) m_1, x_2 is a zero of multiplicity (order) m_2, and so on.

The polynomial function has $\boxed{m_1 + m_2 + \ldots + m_k = n}$ real zeros (m_1 are coincident (same or identical) and equal to x_1, m_2 are coincident and equal to x_2, and so on).

Notes:

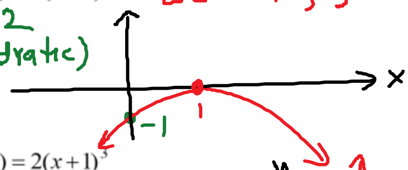
- If m_1 is odd, the function *changes sign* at $x = x_1$ and the graph *crosses* the x-axis.
- If m_1 is even, the function *does not change sign* at $x = x_1$ and the graph *touches* the x-axis.
- If the multiplicity m_1 is greater that 1 then the tangent line at $x = x_1$ is *horizontal*.



Ex 2. Sketch the graph of the following polynomial functions.

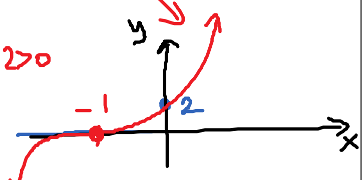
a) $f(x) = -(x-1)^2$

$x_1 = 1$; $m_1 = 2$ (even)
LC = −1 ; y-int = −1
n = 2 (quadratic)



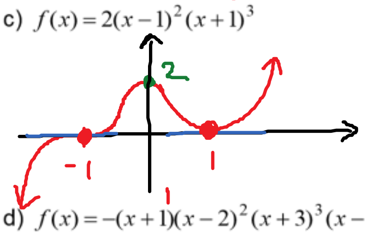
b) $f(x) = 2(x+1)^3$

$x_1 = -1$
$m_1 = 3$
n = 3 (cubic)
LC = 2 > 0



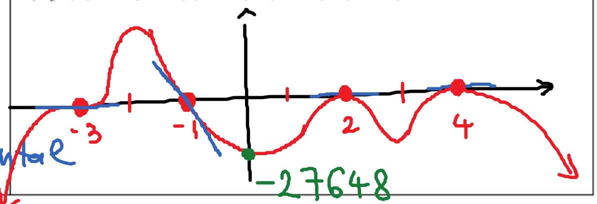
c) $f(x) = 2(x-1)^2(x+1)^3$

$x_1 = 1$; $m_1 = 2$ (even)
$x_2 = -1$; $m_2 = 3$ (odd)
LC = 2 > 0



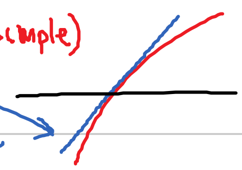
d) $f(x) = -(x+1)(x-2)^2(x+3)^3(x-4)^4$

LC = −1 < 0



$m_1 = 1$ (simple)

TL is oblique



C Non-real Zeros (optional)

A polynomial functions with *non-real zeros* (complex zeros) can be factored as

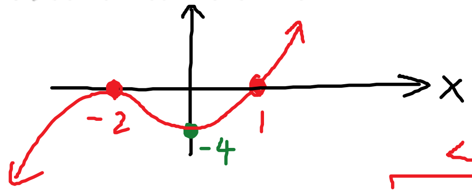
$$f(x) = (a_1x^2 + b_1x + c)^{m^1} \times \ldots$$

where $\Delta_1 = b_1^2 - 4a_1c < 0, \ldots$

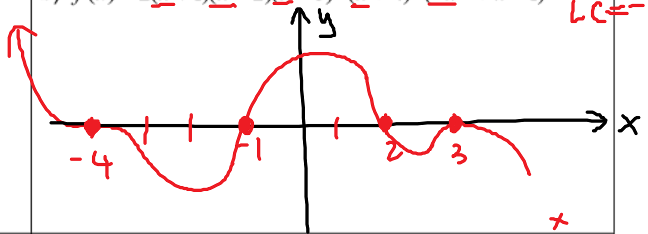
Note. Each trinomial $a_1x^2 + b_1x + c$ has the same sign (the sign of c) for all real numbers x.

Ex 3. Sketch the graph of the following polynomial functions.

a) $f(x) = (x-1)(x+2)^2(x^2+1)$ $LC = 1 > 0$



b) $f(x) = 2(x+1)(x-2)(x-3)^2(x+4)^3(-x^2+x-1)$ $LC = -2$



Ex 4. Find a polynomial $P(x)$ of degree ~~four~~ six with zeros:

- $x_1 = 1$ of multiplicity $m_1 = 3$
- $\boxed{x_2 = -2}$ of multiplicity $\boxed{m_2 = 2}$
- $x_3 = -1$ of multiplicity $m_3 = 1$

such that its graph passes through the point $(2, -8)$. $y = P(x)$

$P(x) = a(x-1)^3(x+2)^2(x+1)$

$P(2) = -8$

$a(2-1)^3(2+2)^2(2+1) = -8$

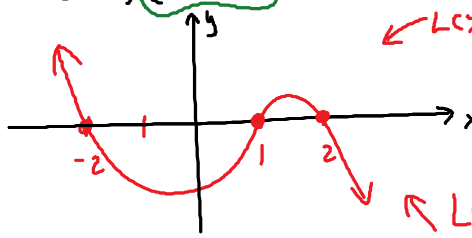
$a = \frac{-8}{(1)(16)(3)} = -\frac{1}{6}$ $\therefore P(x) = -\frac{1}{6}(x-1)^3(x+2)^2(x+1)$

Ex 5. Sketch the graph of the polynomial function:

$y = f(x) = (1-x^3)(x^2-4)$. $n = 5$

$= -(x^3-1)(x^2-4)$

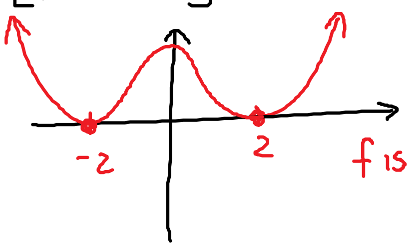
$= -(x-1)(x^2+x+1)(x-2)(x+2)$



Ex 6. Sketch the graph of the polynomial function:

$y = f(x) = (x^2-4)^2$.

$= [(x-2)(x+2)]^2 = (x-2)^2(x+2)^2$

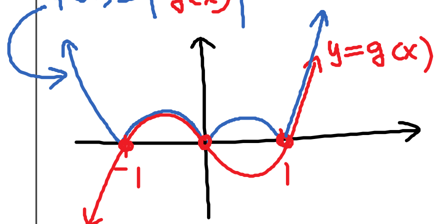


Ex 7. Sketch the graph of the polynomial function:

$y = f(x) = |x(x^2-1)|$.

$g(x) = x(x-1)(x+1)$

$f(x) = |g(x)|$



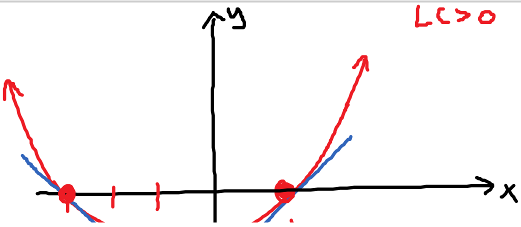
Reading: Nelson Textbook, Pages 139-145

Homework: Nelson Textbook, Page 146: #1, 2, 4, 7, 9ab, 10cd, 13b, 15

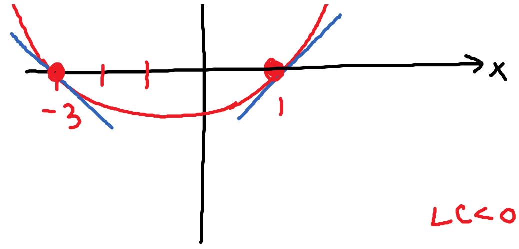
(1a) $f(x) = (x-1)(x+3)$

$= (x-x_1)(x-x_2)$

$x_1 = 1$
$x_2 = -3$ } two simple zeros



$x_1 = 1$
$x_2 = -3$ } two simple zeros



$n = 2$ (quadratic)

$LC = 1 > 0$

$y\text{-int} = -3$

LC<0

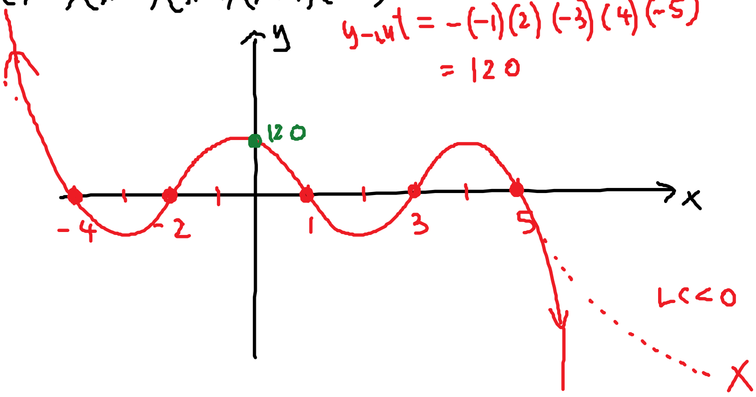
(1d) $f(x) = -(x-1)(x+2)(x-3)(x+4)(x-5)$

$n = 5$

$LC = -1 < 0$

$y\text{-int} = +120$

$y\text{-int} = -(-1)(2)(-3)(4)(-5)$
$= 120$



LC>0

(1e) $f(x) = x^3 - x^2 - 2x$

$= x(x^2 - x - 2)$

$= x(x-2)(x+1)$

$LC = 1 > 0$

$y\text{-int} = 0$