3.2 Characteristics of Polynomial Functions

A End Behaviour

For \( |x| \) very large (\( x \rightarrow \pm \infty \)), the graph of the polynomial function resembles the graph of the leading term \( y = a_n x^n \).

Ex 1. For each case, find an example and sketch the end behaviour of the polynomial function.

1a) \( a_n > 0 \), even \( n \)

\[ f(x) = 2x^4 - 3x^2 + 1 \]

As \( x \rightarrow \pm \infty \), \( y \rightarrow \infty \)

1b) \( a_n > 0 \), odd \( n \)

\[ f(x) = -2x^3 + 3x - 1 \]

As \( x \rightarrow -\infty \), \( y \rightarrow -\infty \)

1c) \( a_n < 0 \), even \( n \)

\[ f(x) = -x^4 + 2x^2 + 1 \]

As \( x \rightarrow \pm \infty \), \( y \rightarrow \infty \)

1d) \( a_n < 0 \), odd \( n \)

\[ f(x) = x^3 - 3x + 1 \]

As \( x \rightarrow -\infty \), \( y \rightarrow -\infty \)

B Symmetry

A polynomial function is even if \( f(-x) = f(x) \). In this case:
- all exponents of \( x \) are even
- graph is symmetric with respect to the y-axis

A polynomial function is odd if \( f(-x) = -f(x) \). In this case:
- all exponents of \( x \) are odd
- graph is symmetric with respect to the origin

Ex 2. Classify as even, odd, or neither.

a) \( f(x) = x^3 + 2x^5 \) odd: \( 1, 3, 5 \) are all odd numbers
b) \( f(x) = 2 - x^2 + 3x^5 \) even: \( 0, 2, 6 \) are all even numbers

c) \( f(x) = 1 + x - x^3 + 3x^4 \) odd: some exponents are odd

d) \( f(x) = (x^2 - 1)^2 \) even: \( x^2 - 1 \) is even

e) \( f(x) = -x(x^2 + 1)^3 \) odd: \( x \) is odd

C Zeros versus x-intercepts

A zero \( z \) is a number (real or complex) where the value of the function is zero.
So \( y = f(z) = 0 \).

An x-intercept \( z - int \) is a real number where the graph of the function intersects the x-axis.
So \( y = f(x - int) = 0 \).

Notes:
- A real zero is also an x-intercept.
- A complex zero is not an x-intercept.

Ex 3. (Optional) Find the zeros and the x-intercepts for \( f(x) = (x - 1)(x^2 + 1) \)

\[ f(x) = (x - 1)(x^2 + 1) = 0 \]

\( x = 1 \) or \( x = \pm i \)

Zeros: \( 1, \pm i \)

x-intercepts: \( 1 \)
D Fundamental Theorem of Algebra
A polynomial function $y = P(x)$ of degree $n$ has $n$ zeros (real or complex).

Notes:
- If the coefficients of the polynomial function are real numbers then complex zeros come in conjugate pairs (the number of complex zeros may be 0, none, 2, 4, 6, ...).
- The number of real zeros is at most $n$. These zeros may be distinct (different) or coincident (same).
- A polynomial function of even degree may have no real zero (all may be complex).
- A polynomial function of odd degree must have at least one real zero $(x - 1)(x - 2)$.

Ex 4. Analyze the general shape for the following polynomial functions:
- a) linear
- b) quadratic
- c) cubic
- d) quartic

E Turning Points
A turning point is the point where an increasing behavior changes to decreasing behavior or vice versa.
A polynomial function of degree $n$ has at most $n-1$ turning points.
A turning point may be either a maximum or a minimum point.

Ex 5. Find the turning points for the polynomial function represented below. What degree may have this polynomial function? Is this degree odd or even?

F Extrema Points
Extrema is the plural of extremum. Extremum is either a minimum or a maximum. An extremum may be local (relative) or global (absolute).
- Each turning point is a local extremum point.
- A polynomial function of even degree has either a global (absolute) minimum or a global (absolute) maximum point.
- A polynomial function of odd degree has neither a global (absolute) minimum nor a global (absolute) maximum point.

Ex 6. Find the local (relative) and global (absolute) extremum points for the polynomial function at example 5.

Ex 7. Find the local (relative) and global (absolute) extremum points for the polynomial function represented below.

Reading: Nelson Textbook, Pages 129-135
Homework: Nelson Textbook, Page 138: #1, 3, 5, 7, 10, 11, 14, 16

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If \( n \) is odd \( n = 1, 3, 5, \ldots \)

4a) \( n = 1 \) (linear)
    One x-cut (real zero)

4b) \( n = 2 \) (quadratic)
    Shape parabola
    2 real zeros
    2 x-cuts (distinct)

4c) cubic \( n = 3 \)
    3 real zeros
    2 complex zeros
    Shape S

4d) \( n = 4 \) (quartic)
    Shape W shape

Example
\[ f(x) = x^4 \]

Turning Point

Local maximum point

Local minimum point

\# of turning points \( \leq n - 1 \)

Example
\[ f(x) = x^2, \quad n = 2 \]
One turning point

Example
\[ f(x) = x^4, \quad n = 4 \]
5. If the leading term of the polynomial function $p(x)$ is $-2x^2$ and the leading term of the polynomial function $q(x)$ is $x^3$, find the leading term of the polynomial function $f(x) = (p(x) + q(x))^2 - p^3(x)$.

\[
\begin{align*}
\left(-2x^2 + x^3\right)^2 & \quad \left(-2x^2 + 2x^3\right)^3 \\
\downarrow & \\
\chi^6 - (-8x^6) & \Rightarrow 9\chi^6
\end{align*}
\]