3.2 Characteristics of Polynomial Functions

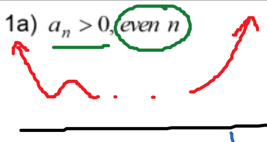
$f(x) = a_n x^n + \cancel{\ldots} \simeq a_n x^n$

A End Behaviour

For $|x|$ very large ($x \to \infty$ or $x \to -\infty$) the graph of the polynomial function resemble the graph of the leading term $y = a_n x^n$.

Ex 1. For each case, find an example and sketch the *end behaviour* of the polynomial function.

1a) $a_n > 0$, even n



Ex $f(x) = 2x^6 - \cancel{x^5} + \cancel{\ldots}$

As $x \to \pm\infty$
$y \to \infty$

$x \to \infty,\ y \to 2\infty^6 = \infty$

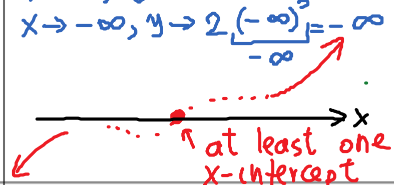
$x \to -\infty,\ y \to 2\underbrace{(-\infty)^6}_{\infty} = \infty$

1b) $a_n > 0$, odd n

Ex $y = \underbrace{2x^3}\ \cancel{-x^2} \cancel{+x}$

$x \to \infty,\ y \to 2(\infty^3) = \infty$

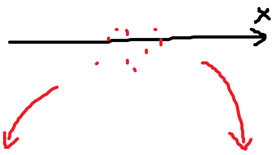
$x \to -\infty,\ y \to 2\underbrace{(-\infty)^3}_{-\infty} = -\infty$



at least one x-intercept

1c) $a_n < 0$, even n

As $x \to \pm\infty$
$y \to -\infty$



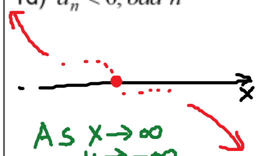
Ex $y = -3x^4 +$

$x \to \infty,\ y \to -3(\infty^4) = -\infty$

$x \to -\infty,\ y \to -3(-\infty)^4 = -3(\infty) = -\infty$

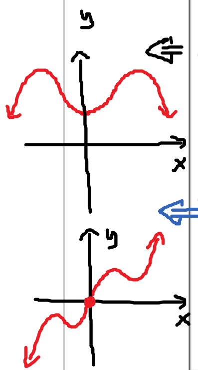
1d) $a_n < 0$, odd n

Ex $y = \underbrace{-x^3}\ \cancel{+x^2} \cancel{+1}$



As $x \to \infty$
$y \to -\infty$
As $x \to -\infty$
$y \to \infty$

B Symmetry



A polynomial function is even if $\boxed{f(-x) = f(x)}$. In this case:

- all exponents of x are even
- graph is symmetric with respect to the y-axis

A polynomial function is odd if $\boxed{f(-x) = -f(x)}$. In this case:

- all exponents of x are odd
- graph is symmetric with respect to the origin

Ex 2. Classify as even, odd, or neither

a) $f(x) = x^1 - x^3 + 2x^5$ odd : 1, 3, 5 are all odd numbers

b) $f(x) = 2 - x^2 + 3x^6$ $2 = 2x^0$, 0, 2, 6 are all even #s
even

c) $f(x) = 1 + x - x^3 + 3x^4$, some exponents are odd and some are even
neither

d) $f(x) = (x^2 - 1)^3$
$f(-x) = [(-x)^2 - 1]^3 = (x^2 - 1)^3 = f(x) \Rightarrow \therefore f$ is even

e) $f(x) = -x^3(x^2 + 1)^2$
$f(-x) = -(-x)^3[(-x)^2 + 1]^2 = x^3(x^2+1)^2 = -f(x)$
$\therefore f$ is odd

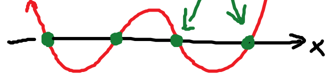
C Zeros versus x-intercepts

$i^2 = -1$

A *zero* z is a number (real or complex) where the *value* of the function is zero.
So $y = \boxed{f(z) = 0}$.

An *x-intercept* $x-\text{int}$ is a *real number* where the graph of the function intersects the x-axis.
So $y = \boxed{f(x-\text{int}) = 0}$.

x-intercepts



Notes.

- A real zeros is also an x-intercepts.
- A complex zero is not an x-intercept.

Ex 3. (Optional) Find the zeros and the x-intercepts for $f(x) = (x-1)(x^2+1)$

$i^2 + 1 = 0$

$f(x) = (x-1)(x^2+1) = 0$

$x - 1 = 0$ or $x^2 + 1 = 0$

$x = 1$ or $x = \pm i$

Zeros: $1, \pm i$

x-int: 1

D Fundamental Theorem of Algebra

A polynomial function $y = P(x)$ of degree n has n zeros (real or complex).

Notes:

- If the *coefficients* of the polynomial function are *real* numbers then complex zeros come in conjugate pairs (the number of complex zeros may be 0 (none), 2, 4, 6, …)
- The number of *real zeros* is *at most n*. These zeros may be distinct (different) or coincident (same)
- A polynomial function of *even* degree *may have no real zero* (all may be complex).
- A polynomial function of *odd* degree must have *at least one* real zero. (x-int)

Ex 4.

Ex 4. Analyze the general shape for the following polynomial functions.

see the solutions below

a) linear

b) quadratic

c) cubic

d) quartic

E Turning Points

A turning point is the point where an increasing behavior changes to decreasing behavior or vice versa.

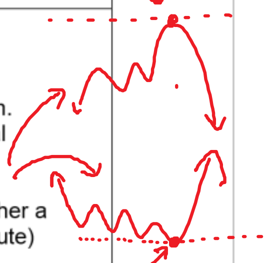
A polynomial function of degree n has at most $n-1$ turning points. # TP ≤ n-1

A turning point may be either a *maximum* or a *minimum* point.

F Extrema Points

Extrema is the plural of extremum.
Extremum is either a minimum or a maximum.
An extremum may be local (relative) or global (absolute).
Each turning point is a local extremum point.
A polynomial function of even degree has either a global (absolute) minimum or a global (absolute) maximum point.
A polynomial function of odd degree has neither a global (absolute) minimum nor a global (absolute) maximum point.

global maximum



global minimum

⇒ no global extremum

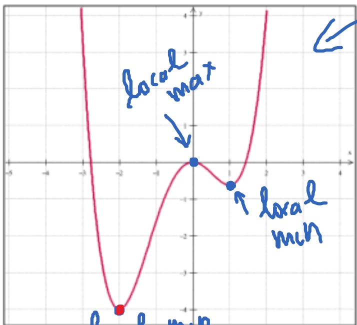
Ex 5.

Ex 5. Find the turning points for the polynomial function represented bellow. What degree may have this polynomial function? Is this degree odd or even?

3 TP

n = 4, 6, 8,

even



local max

local min

local min

global

(-2, -4)

Ex 6.

Ex 6. Find the local (relative) and global (absolute) extremum points for the polynomial function at example 5.

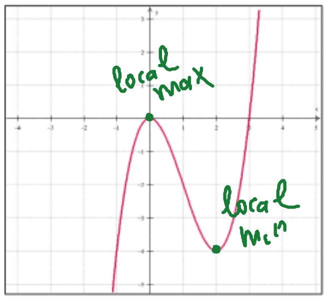
Ex 7.

Ex 7. Find the local (relative) and global (absolute) extremum points for the polynomial function represented bellow.

∴ no global extremum

n = 3, 5, …

$a_n > 0$



local max

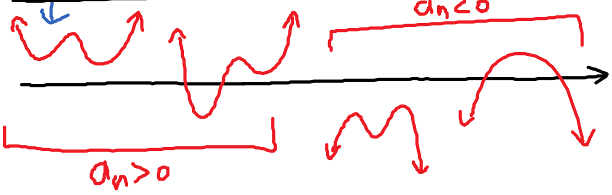
local min

Reading: Nelson Textbook, Pages 129-135
Homework: Nelson Textbook, Page 136: #1, 3, 5, 7, 10, 11, 14, 16

f may have no x-intercept(s)

Conclusions If n is even n = 2, 4, 6, …

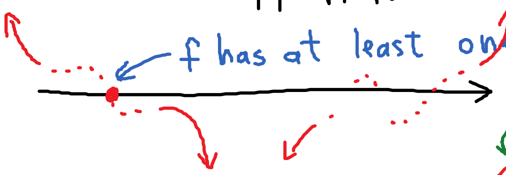
$a_n < 0$



$a_n > 0$

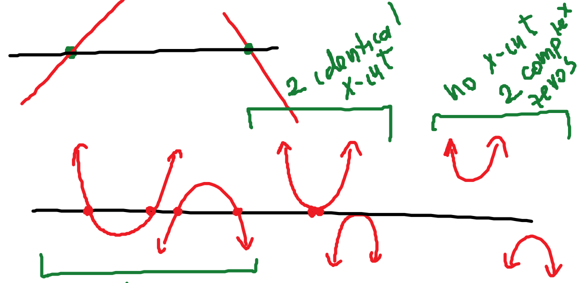
If n is odd $n = 1, 3, 5, \ldots$

f has at least one x-intercept(s)



4a) $n=1$ (linear)
one x-cut
(real zero)

Shape Line



2 identical x-cut

no x-cut
2 complex zeros

4b) $n=2$ (quadratic)

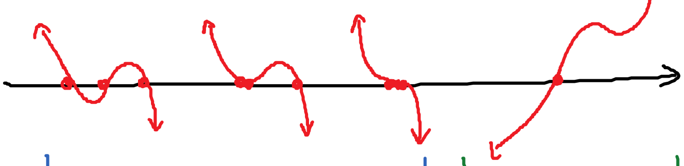
Shape parabola

2 real zeros
2 x-cut
(distinct)

4c) cubic
$n=3$

3 real
0 complex

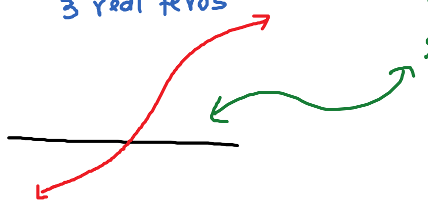
1 real
2 complex



3 real zeros

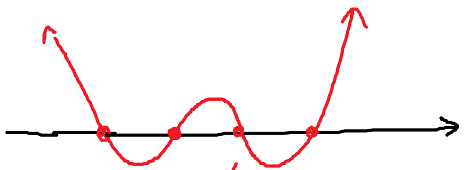
1 real
2 complex

Shape N S shape

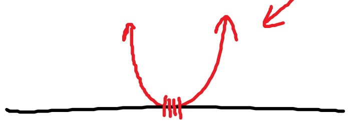


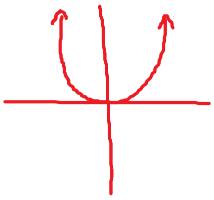
4d) $n=4$ (quartic)

Shape W shape



Ex. $f(x) = x^4$





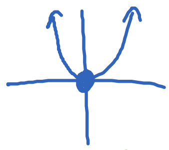
E Turning Point

local maximum point

local minimum point

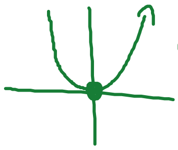
$\boxed{\#\text{ of turning points} \leq n-1}$

Ex $f(x) = x^2$, $n=2$



one turning point

Ex $f(x) = x^4$; $n=4$



one turning point

5. If the leading term of the polynomial function $p(x)$ is $-2x^2$ and the leading term of the polynomial function $q(x)$ is x^3, find the leading term of the polynomial function $f(x) = (p(x) + q(x))^2 - p^3(x)$.

$(-2x^2 + x^3)^2 \quad (-2x^2 + \ )^3$

$x^6 - (-8x^6) \rightarrow 9x^6$