2.1 Determining Average Rate of Change

A Average Rate of Change

\( y = f(x) \), \( y_1 = f(x_1) \), \( y_2 = f(x_2) \)
\( \Delta x = x_2 - x_1 \) (change in variable \( x \))
\( \Delta y = y_2 - y_1 \) (change in variable \( y \))

The Average Rate of Change (ARC) in the \( y \) variable with respect to the \( x \) variable, on (over) the interval \([x_1, x_2]\) (or \( x_1 \leq x \leq x_2 \)) is given by:

\[
ARC = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = m_S
\]

Note. The unit of ARC is:

\[
\text{unit}(ARC) = \frac{\text{unit}(\Delta y)}{\text{unit}(\Delta x)}
\]

Note: The Average Rate of Change (ARC) is equal to the slope of the secant line (\( m_S \)) passing through the points \( P(x_1, y_1) \) and \( Q(x_2, y_2) \).

Ex 1. A rock is launched vertically upward. The height \( h \) (in meters) at the time \( t \) (in seconds) of the rock is given by \( h(t) = 100t - 10t^2 \). Find the average velocity (ARC) over the third second of motion.

\[
t_1 = 2, \quad h_1 = 100(2) - 10(2^2) = 160
\]
\[
t_2 = 3, \quad h_2 = 100(3) - 10(3^2) = 210
\]
\[
AV = \frac{h_2 - h_1}{t_2 - t_1} = \frac{210 - 160}{3 - 2} = 50
\]

\( \therefore \) The average velocity over the third second is 50 m/s

B Secant Line

Let \( y = f(x) \) be a function and \( P(x_1, y_1) \) and \( Q(x_2, y_2) \) two points on its graph.

The slope of the secant line (\( m_S \)) that passes through the points \( P \) and \( Q \) is given by:

\[
m_S = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = ARC
\]

Ex 2. In the figure below is represented the position \( h \) (in kilometers) at the time \( t \) (in hours) of a balloon. Describe the motion of the balloon in terms of average velocity.

\[
AV = \frac{2 - 0}{1 - 0} = 2 \text{ km/h}
\]
\[
AV = \frac{0 - 2}{6 - 4} = -1 \text{ km/h}
\]

\( \therefore \) The balloon is moving up at 2 km/h for one hour, then is not moving for 3 hours and finally is moving down at 1 km/h for two hours.
C Increasing Functions

A function \( f \) is increasing over the interval \((a, b)\) if
\[
ARC = m_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \text{ for all } x_1, x_2 \text{ in the interval } (a, b).
\]

Ex 3. Prove that the function \( y = f(x) = 10^x \) is increasing over its domain.

\[ b = 10 > 1 \]
\[ \therefore \text{ increasing} \]

Ex 5. During an experiment the number of bacteria is measured every minutes (for ten minutes) and the results are presented below:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>200</td>
</tr>
<tr>
<td>2</td>
<td>400</td>
</tr>
<tr>
<td>3</td>
<td>800</td>
</tr>
<tr>
<td>4</td>
<td>1600</td>
</tr>
<tr>
<td>5</td>
<td>3200</td>
</tr>
<tr>
<td>6</td>
<td>6400</td>
</tr>
<tr>
<td>7</td>
<td>12800</td>
</tr>
<tr>
<td>8</td>
<td>25600</td>
</tr>
<tr>
<td>9</td>
<td>51200</td>
</tr>
<tr>
<td>10</td>
<td>102400</td>
</tr>
</tbody>
</table>

\( \text{ARC}_1 = \frac{400 - 100}{2 - 0} = 150 \text{ bacteria/minute} \)
\( \text{ARC}_2 = \frac{102400 - 25600}{10 - 8} = 38400 \text{ bacteria/minute} \)

Compare the average rate of change during the first two minutes and the average rate of change during the last two minutes of the experiment.

D Decreasing Functions

A function \( f \) is decreasing over the interval \((a, b)\) if
\[
ARC = m_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0 \text{ for all } x_1, x_2 \text{ in the interval } (a, b).
\]

Ex 4. Prove that the average rate of change is constant for a linear function:
\[
\text{for } f(x) = \alpha x + b \Rightarrow f(x) = \alpha x_1 + b
\]
\[
y_1 = \alpha x_1 + b
\]
\[
y_2 = \alpha x_2 + b
\]
\[
y_2 - y_1 = \alpha (x_2 - x_1)
\]
\[
ARC = \frac{y_2 - y_1}{x_2 - x_1} = \alpha = \text{constant}
\]

Ex 6. For a given function, the average rate of change over \([2, 4]\) is 5 and the average rate of change over \([4, 7]\) is -2. Find the average rate of change over \([2, 7]\).

\[
\Delta y = 5(2) = 10
\]
\[
\Delta y = (-2)(3) = -6
\]
\[
\text{ARC} = \frac{10 + (-6)}{2 + 3} = \frac{4}{5}
\]

Reading: Nelson Textbook, Pages 68-75
Homework: Nelson Textbook, Page 76: #4, 8, 10

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