Questions 1-5 are Multiple-Choice questions

1. Which of the following relations is INCORRECT
   A) \((x^3)' = 3x^2\)
   B) \((2\sqrt{x})' = x^{-1/2}\)
   C) \(\left(\frac{1}{x^2}\right)' = -\frac{1}{x^3}\)
   D) \(\left(\frac{1}{\sqrt{x}}\right)' = -\frac{1}{3x\sqrt{x}}\)

2. The equation of the tangent line to the curve \(y = x^2\) at the point \(P(1,1)\) is
   A) \(y = -x + 2\)
   B) \(y = 2x - 1\)
   C) \(y = -2x - 1\)
   D) \(y = x + 1\)

3. The derivative function of the function \(f(x) = \frac{x-1}{x+1}\) is
   A) \(f'(x) = -\frac{2x}{x^2 + 1}\)
   B) \(f'(x) = -\frac{x}{(x+1)^2}\)
   C) \(f'(x) = -\frac{2}{x(x+1)^2}\)
   D) \(f'(x) = \frac{2x}{(x+1)^2}\)

4. A function \(f(x) = \frac{1}{x}\) is not differentiable at \(x = 0\) because:
   A) function \(f\) has a vertical tangent at \(x = 0\)
   B) function \(f\) has a removable discontinuity at \(x = 0\)
   C) function \(f\) is not defined at \(x = 0\)
   D) function \(f\) has a horizontal tangent at \(x = 0\)

5. If \(f'(x) = -3x^2 - 2x\) then a possible function \(f(x)\) is
   A) \(f(x) = x^3 - x^2 + x\)
   B) \(f(x) = -6x - 2\)
   C) \(f(x) = -x^3 - x^2\)
   D) \(f(x) = -x^3 + x^2 + x\)

6. For each case, identify the function \(f\) and its derivatives \(f'\) and \(f''\).

Questions 7-15 a long answer questions. Show your work.

7. At what points does the curve \(y = 2x^3 + 3x^2 - 12x - 1\) have a horizontal tangent?

\[
\frac{dy}{dx} = 0 \quad \text{at} \quad y = 6x^2 + 6x - 12
\]

\[
6(x^2 + x - 2) = 0
\]

\[
(x + 2)(x - 1) = 0
\]

\(x = -2 \quad \text{or} \quad x = 1\)

At \(x = -2\)

\(y = 2(-8) + 3(4) + 12(-2) - 1 = -16 + 24 - 1 = 17\)

At \(x = 1\)

\(y = 2 + 3 - 12 - 1 = -8\)

The tangent line is horizontal at \(A(-2, 17)\) or \(B(1, -8)\).
8. For each case, find \( f'(x) \).

\[ \text{[K/U 6 marks]} \]

[2] a) \( f(x) = 1 - x^3 + \frac{2}{x^4} \)

\[ \therefore f'(x) = -3x^2 - \frac{8}{x^5} \]

[2] b) \( f(x) = (x^2 + x)(x^2 - 1) \)

\[ f'(x) = (2x + 1)(x^2 - 1) + (x^2 + x)(2x) \]
\[ = -2x^4 + 2x^3 + 2x^4 - 2x + 2x^3 - 2x - 1 \]
\[ \therefore f'(x) = 5x^3 + 4x^2 - 2x - 1 \]

[2] c) \( f(x) = (\sqrt{x} - 2x)^3 \)

\[ \therefore f'(x) = 3(\sqrt{x} - 2x)^2 \cdot \left( \frac{1}{2\sqrt{x}} - 2 \right) \]

9. For each case, find \( f'(x) \), \( f''(x) \), and \( f'''(x) \).

\[ \text{[K/U 6 marks]} \]

[3] a) \( f(x) = x^4 - 2x^3 \)

\[ f'(x) = 4x^3 - 6x^2 \]
\[ f''(x) = 12x^2 - 12x \]
\[ f'''(x) = 24x - 12 \]

[3] b) \( f(x) = \frac{3x}{x+1} \)

\[ f'(x) = \frac{3(x+1) - 3x}{(x+1)^2} = \frac{3}{(x+1)^2} = 3(x+1)^{-2} \]
\[ f''(x) = -6(x+1)^{-3} \]
\[ f'''(x) = 18(x+1)^{-4} \]

10. Differentiate

\[ f(x) = x + \sqrt{x^2 + \sqrt{x^2 - x}} \]

\[ \therefore f'(x) = 1 + \frac{2x + \frac{2x - 1}{2\sqrt{x^2 + \sqrt{x^2 - x}}}}{2\sqrt{x^2 + \sqrt{x^2 - x}}} \]

11. \( y' = \frac{x-1-x}{(x-1)^2} = \frac{-1}{(x-1)^2} < 0 \) for any \( x \neq 1 \)

The graph is decreasing over \((-\infty, 1)\) or \((1, \infty)\).
11. Show that there are no tangents to the curve \( y = \frac{x}{x-1} \) with positive slope. What can be concluded about the graph?

\[
\frac{dy}{dx} = \frac{x-1-x}{(x-1)^2} = -\frac{1}{(x-1)^2} < 0 \quad \text{for } x \neq 1
\]

The slope \( m \) is \( < 0 \) over \((-\infty, 1) \) or \((1, \infty)\).

\[ \therefore \text{The graph is decreasing over } (-\infty, 1) \text{ or } (1, \infty). \]

12. Find a quadratic function in the form \( f(x) = ax^2 + bx + c \) satisfying the following conditions: \( f(1) = -1 \), \( f'(1) = -3 \), and \( f''(1) = -6 \).

\begin{align*}
f(1) &= -1 \\
&= a + b + c = -1 \\
&\Rightarrow c = -1 - (a - 3) = -3 \\
\Rightarrow f(1) &= -1 - 3 \\
f'(x) &= 2ax + b \\
f'(1) &= 2a + b = -3 \Rightarrow b = -3 - 2(-3) = 3 \\
f''(x) &= 2a \\
f''(1) &= 2a = -6 \Rightarrow a = -3
\end{align*}

\[ \therefore f(x) = -3x^2 + 3x - 1 \quad \therefore f'(x) = -6x + 3 \quad \therefore f''(x) = -6 \]

13. For each case, explain where the function is not differentiable and why.

\[ [A.C. 6 \text{ marks}] \]

2a) \( f(x) = \frac{1}{\sqrt{x}} \)

(1) \( f \) is continuous everywhere.

(2) \( f'(x) = \frac{-1}{2x^{3/2}} \) \( f'(x) \) DNE \( \Rightarrow f \) is not differentiable at \( x = 0 \) (infinite slope)

2b) \( f(x) = |x^2 - 3x + 2| \)

\( f(x) = \begin{cases} x^2 - 3x + 2, & x < 1 \\
-2x + 3, & x \geq 1 \end{cases} \)

\( f'(x) = \begin{cases} 2x - 3, & x < 1 \\
-2, & x \geq 2 \end{cases} \)

\( f'(x) \) DNE \( \Rightarrow f \) is not differentiable at \( x = 1 \) and \( x = 2 \) (corner points)

\( f(x) = \frac{x}{x^2} \)

\( f(x) \) is not defined at \( x = 0 \) and \( x \neq 0 \)

14. Use the first principles to find the derivative of the following function.

\[ f(x) = -3x + \frac{2}{x^2} \]

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-3(x+h) - 2}{(x+h)^2} - \frac{-3x + \frac{2}{x^2}}{h}
\]

\[
= -3 + 2 \lim_{h \to 0} \frac{1}{h} \left[ \frac{1}{(x+h)^2} - \frac{1}{x^2} \right]
\]

\[
= -3 + 2 \lim_{h \to 0} \frac{1}{h} \left[ \frac{x^2 - (x+h)^2}{x^2(x+h)^2} \right]
\]

\[
= -3 + 2 \lim_{h \to 0} \frac{-2x}{x^2(x+h)^2}
\]

\[ f'(x) = -3 - \frac{4}{x^3} \]
15. A position function of a particle is given by \( s(t) = 2t^3 - 6t \).

[a] Find the moments of time when the particle is in origin.

\[ s(t) = 0 \]
\[ 2t(t^2 - 3) = 0 \]
\[ t = 0 \] or \( t = \pm \sqrt{3} \)

\[ t = 0 \] on \( t = \pm \sqrt{3} \)

[1.5] b) Find the velocity function and the moments of time when the particle is at rest.

\[ v(t) = s'(t) = 6t^2 - 6 \]
\[ v(t) = 0 \]
\[ 6t^2 - 6 = 0 \]
\[ t = \pm 1 \]

\[ t^2 = 1 \]

\[ \therefore \] The particle is at rest at \( t = \pm 1 \)

[1.5] c) Find the acceleration function and the moments of time when the acceleration is zero.

\[ a(t) = v'(t) = 12t \]
\[ a(t) = 0 \]
\[ 12t = 0 \]
\[ t = 0 \]

\[ \therefore \] The acceleration is zero at \( t = 0 \)

[1] d) Find intervals of time when the particle is speeding up.

\[ \text{The particle is speeding up over } (-1, 0) \text{ or } (1, \infty) \]

[2] Find the displacement and the total distance travelled over the interval \([-1, 2]\).

\[ s(-1) = -2 + 6 = 4 \]
\[ s(2) = 2(8) - 6(2) = 16 - 12 = 4 \]
\[ \Delta s = s(2) - s(1) = 4 - 4 = 0 \text{ m} \]
\[ \Delta s = 0 \text{ m} \]
\[ \text{The displacement over } [-1, 2] \text{ is } \Delta s = 0 \text{ m} \]

\[ \text{TDT} = \text{Total Distance Traveled} \]
\[ \text{TDT} = 4 + 4 + 4 + 4 = 16 \text{ m} \]

\[ \therefore \] Total Distance travelled over \([-1, 2]\) is 16 m.

[2] d) Sketch the graph of \( s(t) \), \( v(t) \), and \( a(t) \) on the grid provided below.

\[ s(1) = 1 - 6 = -5 \]
\[ s(-1) = 4 \]