1. The function $f$ is defined by the graph represented in the right figure. Find:

a) $\lim_{x \to 4} f(x) = -1$

b) $\lim_{x \to 4} f(x) = 2$

c) $\lim_{x \to 4} f(x) \text{ DNE}$

d) $\lim_{x \to 4} f(x) = -1$

2. Find the numbers $x$ where the function given graphically below is not continuous. Explain what kind (type) of discontinuity is there:

$f$ is not continuous at:

a) $x = -1$ (infinite discontinuity)
   $\lim_{x \to -1} f(x) = \infty$

b) $x = 0$ (jump discontinuity)
   $L \neq \infty$

c) $x = 2$, (point or removable discontinuity)
   $\lim_{x \to 2} f(x) = -1 \neq f(2) \text{ DNE}$

d) $x = 4$
   $-1 = \lim_{x \to 4} \frac{2f(x) - g(x)}{g(x) + 2}$
   (point or removable discontinuity)

3. Given $\lim_{x \to 2} f(x) = -1$ and $\lim_{x \to 2} g(x) = 3$, use the limits properties to find $\lim_{x \to 2} \frac{2f(x) - g(x)}{g(x) + 2}$ [K/U 3 marks]

$$\lim_{x \to 2} \frac{2f(x) - g(x)}{g(x) + 2} = \frac{2 \lim_{x \to 2} f(x) - \lim_{x \to 2} g(x)}{\lim_{x \to 2} g(x) + 2} \quad \text{(1)}$$

$$= \frac{2(-1) - 3}{2 + 2} \quad \text{(4)}$$

$$= -1 \quad \text{(5)}$$

$$\therefore \lim_{x \to 2} \frac{2f(x) - g(x)}{g(x) + 2} = -1$$
4. The function \( y = f(x) \) is given by its graph in the figure below.

[1] a) Find the rate of change in the \( y \) variable over the interval \([0,2]\).

\[
\begin{align*}
X_1 &= 0, & Y_1 &= 0, \\
X_2 &= 2, & Y_2 &= 4, \\
\Delta Y &= 4 - 0, & \Delta X &= 2 - 0, \\
\therefore \frac{\Delta Y}{\Delta X} &= \frac{4}{2} = 2.
\end{align*}
\]

\[
\therefore R \cdot C = 2.
\]

b) Find the slope of the tangent line at the point \( P \) \( (2,4) \).

\[
\begin{align*}
A &= (-4, -1), & B &= (8, 9), \\
\Delta Y &= 9 - (-1) = 10, & \Delta X &= 8 - (-4) = 12, \\
\therefore m &= \frac{\Delta Y}{\Delta X} = \frac{10}{12}, \\
&= \frac{5}{6}.
\end{align*}
\]

\[
\therefore m \approx 0.83.
\]

5. Find each limit.

[1] a) \( \lim_{x \to 2} \frac{x^2 + 1}{x + 1} = \frac{2^2 + 1}{2 + 1} = \frac{5}{3} = \frac{5}{3} \)

\[
\therefore \lim_{x \to 2} \frac{x^2 + 1}{x + 1} = 3.
\]

b) \( \lim_{x \to 3} \frac{\sqrt{x} + 3}{x - 2} \)

\[
\begin{align*}
L &= \lim_{x \to 2+} \frac{\sqrt{x} + 3}{x - 2}, & L &= \text{DNE} \\
R &= \lim_{x \to 2-} \frac{\sqrt{x} + 3}{x - 2}, & R &= 0
\end{align*}
\]

\[
\therefore L \neq R.
\]

\[
\lim_{x \to 3} \frac{\sqrt{x} + 3}{x - 2} = \text{DNE}
\]

[2] c) \( \lim_{x \to 3} \frac{x^2 - 2x + 1}{x^2 - 2x - 2} \)

\[
\begin{align*}
\lim_{x \to 1} \frac{(x - 1)(x - 1)}{x - 1} (x + 2) \\
&= \lim_{x \to 1} \frac{x - 1}{x + 2} \\
&= \frac{1 - 1}{1 + 2} = 0
\end{align*}
\]

[2.5] d) \( \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 4} - \sqrt{4x^2 - 4}} \cdot \sqrt{x + 4} \)

\[
\begin{align*}
\lim_{x \to 16} \frac{\sqrt{x - 16}}{\sqrt{x - 16} (\sqrt{x + 4} - \sqrt{4})} \\
&= \lim_{x \to 16} \frac{1}{\sqrt{x + 4}} \\
&= \frac{1}{\sqrt{16 + 4}} \\
&= \frac{1}{8}
\end{align*}
\]

\[
\therefore \lim_{x \to 16} \frac{\sqrt{x - 16}}{x - 16} = \frac{1}{8}.
\]

[2.5] f) \( \lim_{x \to 2} \frac{x^2 + 8}{x^2 - 4} \)

\[
\begin{align*}
\lim_{x \to 2} \frac{(x + 2)(x^2 - 2x + 4)}{(x - 2)(x + 2)} \\
&= \lim_{x \to 2} \frac{x^2 - 2x + 4}{x - 2} \\
&= \frac{(-2)^2 - 2(-2) + 4}{-2 - 2} \\
&= \frac{10}{-4} = -3
\end{align*}
\]

\[
\therefore \lim_{x \to 2} \frac{x^2 + 8}{x^2 - 4} = -3.
\]
6. Analyse the continuity of the following function. Graph the function. [4 A 4 marks]

\[ f(x) = \begin{cases} \frac{x^3 + 2x}{4-x}, & x \leq 1 \\ \frac{3x + 2}{\sqrt{x-3}}, & x > 1 \end{cases} \]

At \( x = 1 \)

\[ L = \lim_{x \to 1^-} (x^3 + 2x) = 3 \]

\[ R = \lim_{x \to 1^+} (4-x) = 3 \]

\[ f(1) = 1^2 + 2(1) = 3 \]

\[ \therefore f \text{ is continuous at } x = 1 \]

At \( x = 3 \)

\[ L = \lim_{x \to 3^-} \sqrt{x-3} = 0 = f(3) \]

\[ R = \lim_{x \to 3^+} \sqrt{x-3} = 0 \]

\[ \therefore f \text{ is not continuous at } x = 3 \]

\[ \therefore f \text{ is continuous at any number except } x = 3 \text{ (jump discontinuity)} \]

7. Consider the following position function: \( s(t) = 2t^2 - 6t \) [5 A 5 marks]

[1] a) Find the average velocity over the time interval \([0,1]\)

\[ t_1 = 0; \quad s_1 = s(0) = 0 \]

\[ t_2 = 1; \quad s_2 = s(1) = 2 - 6 = -4 \]

\[ \text{AV} = \frac{\Delta s}{\Delta t} = \frac{-4 - 0}{1 - 0} = -4 \quad \therefore \text{The AV over } [0,1] \text{ is } -4 \text{ m/s} \]

[3] b) Find the instantaneous velocity at the generic moment \( t = a \). Show your work.

\[ s(a) = 2a^2 - 6a \]

\[ s(a + h) = 2(a + h)^2 - 6(a + h) \]

\[ v = \lim_{h \to 0} \frac{2(a + h)^2 - 6(a + h) - (2a^2 - 6a)}{h} \]

\[ = \lim_{h \to 0} \frac{2(a + h)^2 - 6a}{h} \]

\[ = \lim_{h \to 0} \frac{2(a + h)^2 - 6a}{h} \]

\[ = \lim_{h \to 0} \frac{2a^2 + 4ah + 2h^2 - 6a}{h} \]

\[ = 2a \]

\[ v = 6a^2 - 6 \quad \therefore \text{The IV at } t = a \text{ is } \]

[1] c) Find the moments when the particle is at rest (velocity is zero).

\[ v = 0 \]

\[ 6(a^2 - 1) = 0 \quad \therefore \text{The particle is at rest when } t = \pm 1 \text{ s} \]

8. Find the equation of the tangent line to the graph of \( y = f(x) = \sqrt{2x - 1} \) at the point \( P(5,3) \). Show your work. [4 A 4 marks]

\[ f(5) = 3 \]

\[ f(5+h) = \sqrt{2(5+h) - 1} \]

\[ = \sqrt{9 + 2h} \]

\[ m = \lim_{h \to 0} \frac{\sqrt{9 + 2h} - 3}{h} \cdot \frac{\sqrt{9 + 2h} + 3}{\sqrt{9 + 2h} + 3} \]

\[ = \lim_{h \to 0} \frac{9 + 2h - 9}{h (\sqrt{9 + 2h} + 3)} \]

\[ = \lim_{h \to 0} \frac{2}{\sqrt{9 + 2h} + 3} \]

\[ = \frac{2}{6} = \frac{1}{3} \]

\[ m = \frac{1}{3} \]

\[ y - 3 = \frac{1}{3} (x - 5) \]

\[ y = \frac{1}{3} x + 3 - \frac{5}{3} \]

\[ y = \frac{1}{3} x + \frac{4}{3} \]

\[ \therefore \text{The equation of the tangent line at } P(5,3) \text{ is } \]

\[ m = \frac{1}{3} (x + 4) \]
9. Analyse the continuity of the function \( y = f(x) = \frac{x^2 - 9}{|x - 3|} \). Graph and explain. [A 5 marks]

\[
f(x) = \begin{cases} 
  x + 3, & x > 3 \\
  -x - 3, & x < 3 
\end{cases}
\]

\( f \) is continuous over \((-\infty, 3) \) or \((3, \infty)\)

At \( x = 3 \)

\[
L = \lim_{x \to 3^-} (x - 3) = -6 \\
R = \lim_{x \to 3^+} (x - 3) = 6
\]

\( f(3) \) DNE

\( f \) is not continuous at \( x = 3 \) (jump discontinuity)

10. Consider the piecewise defined function below. Find the values of the constants \( a \) and \( b \) such that the function \( y = f(x) \) to be continuous at any number. Show your work. [A 4 marks]

\[
f(x) = \begin{cases} 
  \sqrt{x}, & 0 < x < 4 \\
  ax^2 + bx + 1, & x \geq 4
\end{cases}
\]

At \( x = 0 \)

\[
L = \lim_{x \to 0^-} (x + a) = a \\
R = \lim_{x \to 0^+} \sqrt{x} = 0
\]

\( f(0) = 0 + a = a \)

\( f \) is continuous at \( x = 0 \) if \( a = 0 \)

At \( x = 4 \)

\[
L = \lim_{x \to 4^-} \sqrt{x} = \sqrt{4} = 2 \\
R = \lim_{x \to 4^+} (ax^2 + bx + 1) = 16a + 4b + 1 = 4b + 1
\]

\( f(4) = 16a + 4b + 1 = 4b + 1 \)

\( f \) is continuous at \( x = 0 \) and \( x = 4 \) if

\[
a = 0 \\
b = \frac{1}{4}
\]

\( f \) is continuous at any number if \( a = 0 \) and \( b = \frac{1}{4} \)

11. Compute the limit \( \lim_{x \to 1} \frac{x - \sqrt{x}}{\sqrt{x} - 1} \). [TIPS 4 marks]

\[
u = \frac{x}{\sqrt{x}} \\
u^2 = \frac{3}{\sqrt{x}} \\
u^3 = \sqrt{x} \\
x \to 1 \\
u \to 1
\]

\[
\begin{align*}
\frac{u^6 - u^3}{u^2 - 1} &= \lim_{u \to 1} \frac{u^6 - u^3}{u^2 - 1} \\
\frac{u^2(u^4 - 1)}{u^2 - 1} &= \lim_{u \to 1} \frac{u^2(u^4 - 1)}{u^2 - 1} \\
\frac{u^3(u^2 - 1)}{(u - 1)(u + 1)} &= \lim_{u \to 1} \frac{u^3(u^2 - 1)}{(u - 1)(u + 1)} \\
\end{align*}
\]

\( \therefore \lim_{x \to 1} \frac{x - \sqrt{x}}{\sqrt{x} - 1} = \frac{3}{2} \)