1. The function \( f \) is defined by the graph represented in the right figure. Find:

a) \( \lim_{x \to 0^-} f(x) = -1 \)

b) \( \lim_{x \to 0^+} f(x) = 3 \)

c) \( \lim_{x \to 0} f(x) \) DNE

d) \( \lim_{x \to a} f(x) = 3 \)

e) \( \lim_{x \to a^+} f(x) = 1 \)

f) \( \lim_{x \to -1^-} f(x) \) DNE

2. Consider the following function defined by its graph:

Analyze the discontinuity of this function (continuous or discontinuous) and the type of discontinuity (removable, jump or infinite discontinuity) at the following numbers. Justify your answer (explain why).

a) at \( x = -5 \)

Continuous:

\[
\lim_{x \to -5^-} f(x) = \lim_{x \to -5^+} f(x) = 1
\]

b) at \( x = 2 \)

Discontinuous (Jump Discontinuity):

\[
\lim_{x \to 2^-} f(x) = -2 \neq \lim_{x \to 2^+} f(x) = 3
\]

c) at \( x = 3 \)

Discontinuous (Removable Discontinuity):

\[
\lim_{x \to 3} f(x) = 2 \neq f(3) = 1
\]

3. Analyse the limit and the continuity of the signum function \( \text{sgn}(x) \) at \( x = 0 \). Graph and explain:

\[
\text{sgn}(x) = \begin{cases} 
-1 & \text{if } x < 0 \\
0 & \text{if } x = 0 \\
1 & \text{if } x > 0 
\end{cases}
\]

\[
\lim_{x \to 0^-} \text{sgn}(x) = -1
\]

\[
\lim_{x \to 0^+} \text{sgn}(x) = 1
\]

\[\text{sgn}(0) = 0\]

\(\text{sgn}(x)\) is discontinuous at \( x = 0 \)

There is a jump discontinuity at \( x = 0 \) because

\[
\lim_{x \to 0^-} \text{sgn}(x) = -1 \neq 1 = \lim_{x \to 0^+} \text{sgn}(x)
\]
4. Find each limit.

\[ \lim_{x \to 0} \frac{x^2 - 9}{x + 3} = \frac{-9}{3} = -3 \]

\[ \lim_{x \to 3} \frac{x^2 - 6x + 8}{x^2 - 4x - 8} = \lim_{x \to 3} \frac{(x-4)(x-2)}{(x-4)(x+2)} = \lim_{x \to 4} \frac{x-2}{x+2} = \frac{4-2}{4+2} = \frac{2}{6} = \frac{1}{3} \]

\[ \lim_{x \to 3} \frac{x^3 + 27}{x^2 - 9} = \lim_{x \to 3} \frac{(x+3)(x^2 - 3x + 9)}{(x-3)(x+3)} = \lim_{x \to 3} \frac{x^2 - 3x + 9}{x-3} = \frac{9 + 9 + 9}{-3 - 3} = -\frac{27}{6} = -\frac{9}{2} \]

\[ \lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} = \lim_{x \to 0} \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} = \frac{\frac{1}{2}\sqrt{1+x} + \frac{1}{2}\sqrt{1-x}}{\frac{1}{2}\sqrt{1+x} + \frac{1}{2}\sqrt{1-x}} = \frac{1}{2} \]

\[ \lim_{x \to 3/2} \frac{x^{3/2} - 1}{x - 1} = \lim_{x \to 3/2} \frac{\frac{3}{2}x^{1/2} - 1}{1/2} = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4} \]

\[ \lim_{t \to 0} \frac{1 - \frac{1}{1+t^2} + \frac{1}{1+t}}{1 - \frac{1}{1+t^2} + \frac{1}{1+t}} = \lim_{t \to 0} \frac{1 - \frac{1}{1+t} - \frac{1}{1+t^2}}{1 - \frac{1}{1+t} - \frac{1}{1+t^2}} = -\frac{1}{2} \]

5. Consider the piecewise defined function below. Find the values of the constants \( a \) and \( b \) such that the function \( y = f(x) \) to be continuous at any number. Show your work.

\[
\begin{align*}
  f(x) &= \begin{cases} 
    3 & x < 1 \\
    \sqrt{x + b} & x \geq 1 
  \end{cases}
\end{align*}
\]

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left[ \frac{3}{x + 2} - a \right] = 1 - a \]

\[ 1 - a = 2 \Rightarrow a = -1 \]

\[ f(1) = 2 \]

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left[ \sqrt{x + b} \right] = \sqrt{1 + b} \]

\[ 1 = \sqrt{1 + b} \Rightarrow y = \sqrt{1 + b} \Rightarrow b = 3 \]

\[ f \] is continuous at any number if \( a = -1 \) and \( b = 3 \]
6. Consider the following position function: \( s(t) = 2t^2 - 1 \)

[1] a) Find the average velocity over the time interval [5, 10].

\[ t_1 = 5; \quad s_1 = s(5) = 2(5)^2 - 1 = 49 \]
\[ t_2 = 10; \quad s_2 = s(10) = 2(10)^2 - 1 = 199 \]

\[ \text{AV} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{199 - 49}{10 - 5} = \frac{150}{5} = 30 \text{ m/s} \]

[5,10] is 0.4 m/s

b) Find the instantaneous velocity at \( t = 5 \). Show your work.

\[ v = \lim_{h \to 0} \frac{s(5+h) - s(5)}{h} = \lim_{h \to 0} \frac{2(5+h)^2 - 4}{2(5+h) - 4} \]
\[ = \lim_{h \to 0} \frac{4(15+10h+h^2) - 16}{2(5+h) - 4} = \lim_{h \to 0} \frac{4 \cdot \frac{h}{2} + \frac{h^2}{2}}{4} = \frac{1}{2} \]

The instantaneous velocity at \( t = 5 \) is 0.5 m/s

7. Find the equation of the tangent line to the graph of \( f(x) = 3x - \frac{2}{x} \) at the point \( P(2,5) \). Show your work.

\[ m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{3(2+h) - 2}{2+h} = \lim_{h \to 0} \frac{6h + (3h^2 + 2h) - 2}{2+h} = \lim_{h \to 0} \frac{6h + (7h + 3h^2)}{2} = \frac{3}{2} \]

\[ y - y_1 = m(x - x_1) \]
\[ y - 5 = \frac{3}{2}(x - 2) \]
\[ y = \frac{3}{2}x - 2 \]

The equation of the tangent line is:

8. Consider the function: \( f(x) = -3x^2 + 2x \)

[3] a) Use the alternate formula \( m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \) to find the slope of the tangent line to the graph of the curve at the generic point \( P(a, f(a)) \).

\[ m = \lim_{x \to a} \frac{-3x^2 + 2x - (-3a^2 + 2a)}{x - a} = \lim_{x \to a} \frac{-3x^2 - 3a^2 + 2x - 2a}{x - a} \]
\[ = -3 \lim_{x \to a} \frac{(x-a)(2x+2a)}{x - a} + 2 \lim_{x \to a} \frac{(x-a)(2a)}{x - a} \]
\[ = -6a + 6a^2 \]

\[ \therefore m = -6a + 6a^2 \]

b) Find the point(s) where the tangent line is horizontal.

\[ m = 0 \]
\[ -6a + 6a^2 = 0 \]
\[ a = 0 \text{ or } a = 1 \]

\[ a = 0 \text{ or } a = 1 \]

The tangent is horizontal at \( A(0,0) \) or \( B(1,1) \)
9. Analyse the continuity of the function. Graph the function.

\[ f(x) = \begin{cases} \frac{x(x-1)}{x-1}, & x \neq 1 \\ 0, & x = 1 \end{cases} \]

\[ f(x) = \begin{cases} x > x > 1 \\ 0, & x = 1 \\ -x > x < 1 \end{cases} \]

\[ \lim_{x \to 1^-} f(x) = -1 \]
\[ \lim_{x \to 1^+} f(x) = 1 \]

\[ \Rightarrow f(x) \text{ is discontinuous at } x = 1 \]

There is a jump discontinuity at \( x = 1 \).

10. An oil tank is being drained for cleaning. After \( t \) minutes there are \( V \) litres of oil left in the tank, where

\[ V(t) = 35(25 - t)^2, \quad 0 \leq t \leq 25. \]

(a) Determine what \( V(0) \) and \( V(25) \) represent.

\[ V(0) = 25(25 - 0)^2 = 21875 \text{ litres is the initial volume of oil}. \]
\[ V(25) = 25(25 - 25)^2 = 0 \text{ is the final volume of oil (after 25 minutes)}. \]

(b) Determine the average rate of change of volume during the first 15 minutes.

\[ t_1 = 0 \Rightarrow V_1 = V(0) = 21875 \]
\[ t_2 = 15 \Rightarrow V_2 = V(15) = 25(25 - 15)^2 = 3500 \]

\[ \frac{AV}{\Delta t} = \frac{V_2 - V_1}{t_2 - t_1} = \frac{3500 - 21875}{15 - 0} = -1225 \]

\[ \therefore \text{ The average rate of change of volume during the first 15 minutes is } -1225 \text{ litres/minute}. \]

(c) Determine the rate of change of volume at the time \( t = 15 \) minutes.

\[ \text{IRC} = \lim_{h \to 0} \frac{V(15 + h) - V(15)}{h} = 25 \lim_{h \to 0} \frac{(10-h)^2 - 100}{h} \]
\[ = 25 \lim_{h \to 0} \frac{(10-h)^2 - 100}{h} = 25 \lim_{h \to 0} \frac{100 - 20h + h^2 - 100}{h} \]
\[ = 25 \lim_{h \to 0} \frac{h(-20 + h)}{h} = 25 \lim_{h \to 0} (-20 + h) = 25(-20) = -700 \]

\[ \therefore \text{ The instantaneous rate of change in the volume with respect to time at } t = 15 \text{ minutes is } -700 \text{ litres/minute}. \]
11. Use technology (a scientific calculator) to estimate the slope of the tangent line to the curve \( y = \sqrt{x + \sqrt{x}} \) at the point \( (4, \sqrt{6}) \) by using \( h = 0.0001 \). Show your work.  

\[
\begin{align*}
M & = \frac{\sqrt{4 + 0.0001} + \sqrt{4 + 0.0001} - \sqrt{4} + \sqrt{4}}{0.0001} \\
& = \frac{\sqrt{4.0001} + \sqrt{4.0001} - \sqrt{4}}{0.0001} \\
& = 0.2551535
\end{align*}
\]

\( \therefore \text{The slope of tangent at } P (4, \sqrt{6}) \text{ is approximately equal to} 0.255 \)

12. Find \( a \) and \( b \) such that \( \lim_{x \to 0} \frac{\sqrt{ax + b} - 2}{x} = 1 \).  

\[
\begin{align*}
\lim_{x \to 0} f(x) = 1 \quad \text{and} \quad x \to 0 = \lim_{x \to 0} \sqrt{ax + b} - 2 = 0 = \\
\sqrt{b} - 2 = 0 \Rightarrow b = 4
\end{align*}
\]

\[
\begin{align*}
\lim_{x \to 0} \frac{\sqrt{ax + 4} - 2}{x} = \lim_{x \to 0} \frac{\sqrt{ax + 4} - 2}{x} \cdot \frac{\sqrt{ax + 4} + 2}{\sqrt{ax + 4} + 2} = \\
= \lim_{x \to 0} \frac{(ax + 4) - 4}{x(\sqrt{ax + 4} + 2)} = \lim_{x \to 0} \frac{ax}{x(\sqrt{ax + 4} + 2)} = \lim_{x \to 0} \frac{a}{\sqrt{ax + 4} + 2}
\end{align*}
\]

\[
\begin{align*}
\frac{a}{4} = 1 \Rightarrow a = 4
\end{align*}
\]

\( \therefore \lim_{x \to 0} \frac{\sqrt{ax + b} - 2}{x} = 1 \text{ if } a = 4 \text{ and } b = 4 \).