## 9.4 Intersection of three Planes

**A Intersection of three Planes**
Let consider three planes given by their Cartesian equations:

$$\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$$
$$\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$$
$$\pi_3 : A_3x + B_3y + C_3z + D_3 = 0$$

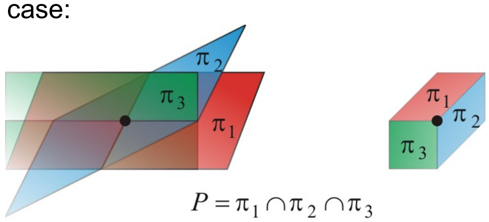
The point(s) of *intersection* of these planes is (are) related to the *solution(s)* of the following system of equations:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \\ A_3x + B_3y + C_3z + D_3 = 0 \end{cases} \quad (*)$$

There are *three* equations and *three* unknowns.
You may use *substitution* or *elimination* to solve this system.

**B Unique Solution**
(Point Intersection – Non Coplanar Normal Vectors)

In this case:



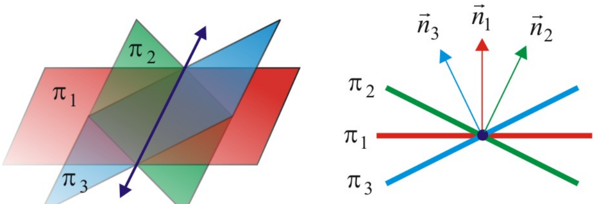

$P = \pi_1 \cap \pi_2 \cap \pi_3$

- ⇨ The planes *intersect* into a *single point*.
- ⇨ The *normal vectors* are *not coplanar*: $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) \neq 0$ .
- ⇨ By solving the system (*), you get a *unique solution* for $x$ , $y$ , and $z$ .

Ex 1. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x - 3y - 2z = -9 & (1) \\ 2x - 5y + z = 3 & (2) \\ -3x + 6y + 2z = 8 & (3) \end{cases}$$

**C Infinite Number of Solutions (I)**
(Line Intersection – Non Parallel Planes and Coplanar Normal Vectors)




$L = \pi_1 \cap \pi_2 \cap \pi_3$

Ex 2. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + y + 2z = -2 \\ 3x - y + 14z = 6 \\ x + 2y = -5 \end{cases}$$

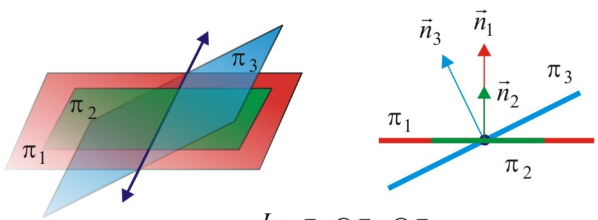

 

In this case:

- ⇨ The planes are *not parallel* but their normal vectors are *coplanar*: $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$.
- ⇨ The intersection is a *line*.
- ⇨ One scalar equation is a *combination* of the other two equations.
- ⇨ By solving the system (*), you may express two variables in terms of the third one using two equations.

**D Infinite Number of Solutions (II)**
(Line Intersection – Two Coincident Planes and one Intersecting Plane)

In this case:




$$L = \pi_1 \cap \pi_2 \cap \pi_3$$

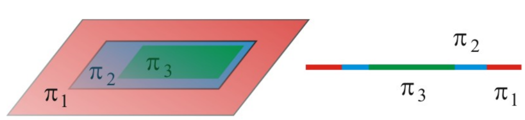
- ⇨ *Two* planes are *coincident* and the third plane is *not parallel* to the coincident planes.
- ⇨ The intersection is a *line*.
- ⇨ Two scalar equations are *equivalent*. The coefficients $A, B, C, D$ are *proportional* for these two equations.
- ⇨ You may express two variables in terms of the third one using two non equivalent equations.

Ex 3. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + y - z = 2 & (1) \\ x - 2y + z = 4 & (2) \\ 2x - 4y + 2z = 8 & (3) \end{cases}$$

**E Infinite Number of Solutions (III)**
(Plane Intersection – Three Coincident Planes)

In this case:




$$\pi_1 \cap \pi_2 \cap \pi_3 = \pi_1 = \pi_2 = \pi_3$$

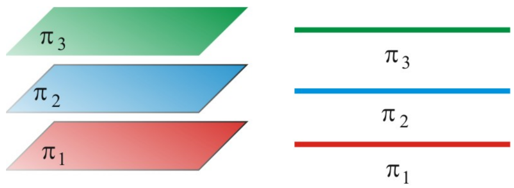
- ⇨ The coefficients $A, B, C, D$ are *proportional* for all three equations.
- ⇨ Any point of one plane is also a point on the other two planes.
- ⇨ The intersection is a *plane*.

Ex 4. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x - y - 2z = 1 \\ 2x - 2y - 4z = 2 \\ -4x + 4y + 8z = -4 \end{cases}$$

**F No Solution**
**(Parallel and Distinct Planes)**

In this case:



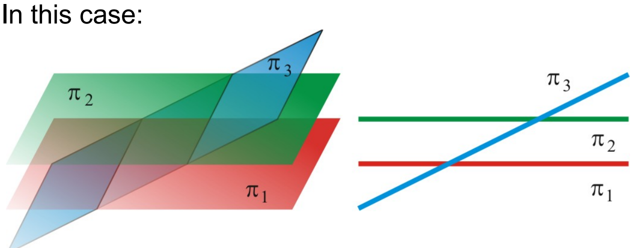

- ⇨ There are three *parallel* and *distinct* planes.
- ⇨ There is *no point of intersection*.
- ⇨ There is *no solution* for the system of equations (the system of equations is *incompatible*).
- ⇨ The coefficients $A, B, C$ are *proportional* but the coefficients $A, B, C, D$ are *not proportional*.
- ⇨ By solving the system (*) you get *false* statements (like $0 = 1$).

Ex 5. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + 2y + 3z = 1 & (1) \\ 2x + 4y + 6z = -1 & (2) \\ -x - 2y - 3z = 3 & (3) \end{cases}$$

**G No Solution (II)**
**(H Configuration)**

In this case:



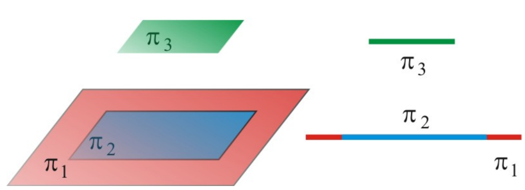

- ⇨ Two planes are *parallel and distinct* and the third plane is *intersecting*.
- ⇨ There is *no point of intersection*.
- ⇨ The coefficients $A, B, C$ are *proportional* for two planes.
- ⇨ There is *no solution* for the system of equations (the system of equations is *incompatible*).
- ⇨ By solving the system (*) you get false statements (like $0 = 1$).

Ex 6. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + y - z = 1 & (1) \\ x + y + z = 2 & (2) \\ -2x - 2y + 2z = 3 & (3) \end{cases}$$

**H No Solution (III)**

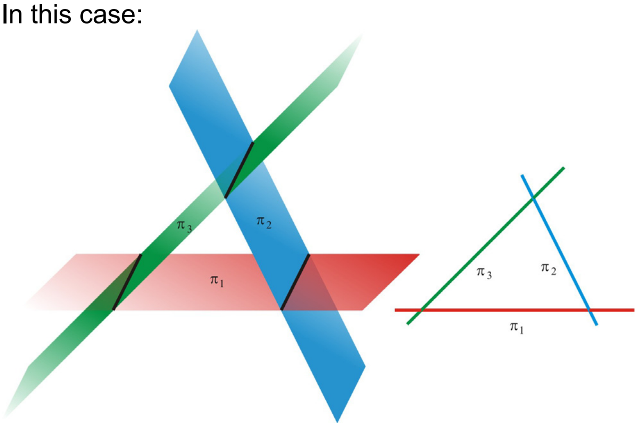
In this case:




- ⇨ Three planes are *parallel* but only *two are*

Ex 7. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + 2y + 3z = 1 & (1) \\ 3x + 6y + 9z = 3 & (2) \\ -2x - 4y - 6z = 2 & (3) \end{cases}$$

| | |
|---|---|
| *coincident*.<br>⇨ The coefficients $A, B, C$ are *proportional* for all equations but the coefficients $A, B, C, D$ are *proportional* only for two planes.<br>⇨ There is *no solution* for the system of equations (the system of equations is *incompatible*).<br>⇨ By solving the system (*) you get *false* statements (like $0 = 1$). | |
| **I No Solution (IV)**<br>(Delta Configuration)<br><br>In this case:<br><br>⇨ The planes are *not parallel* (the coefficients $A, B, C$ are *not proportional*).<br>⇨ The normal vectors are *coplanar* ($\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$).<br>⇨ There is *no point of intersection* between all three planes.<br>⇨ There is *no solution* for the system of equations (the system of equations is *incompatible*).<br>⇨ By solving the system (*) you get *false* statements (like $0 = 1$). | Ex 8. Solve the following system of equations. Give a geometric interpretation of the solution(s).<br>$\begin{cases} 2x + y + z = 1 & (1) \\ -x + y + z = -1 & (2) \\ x + y + z = 0 & (3) \end{cases}$ |

**Reading**: Nelson Textbook, Pages 520-529
**Homework**: Nelson Textbook: Page 530 #8, 9, 10, 13