

9.4 Intersection of three Planes

A Intersection of three Planes

Let consider three planes given by their Cartesian equations:

$$\begin{aligned} \pi_1 : A_1x + B_1y + C_1z + D_1 &= 0 \\ \pi_2 : A_2x + B_2y + C_2z + D_2 &= 0 \\ \pi_3 : A_3x + B_3y + C_3z + D_3 &= 0 \end{aligned}$$

The point(s) of *intersection* of these planes is (are) related to the *solution(s)* of the following system of equations:

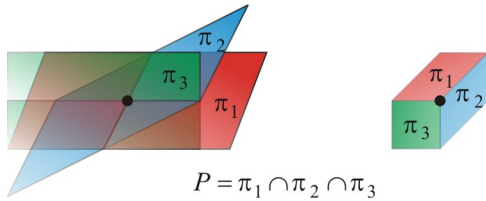
$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \quad (*) \\ A_3x + B_3y + C_3z + D_3 = 0 \end{cases}$$

There are *three* equations and *three* unknowns. You may use *substitution* or *elimination* to solve this system.

B Unique Solution

(Point Intersection – Non Coplanar Normal Vectors)

In this case:



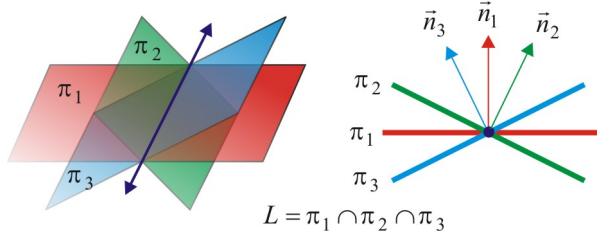
- ⇒ The planes *intersect* into a *single point*.
- ⇒ The *normal vectors* are *not coplanar*.
 $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) \neq 0$.
- ⇒ By solving the system (*), you get a *unique solution* for x , y , and z .

Ex 1. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x - 3y - 2z = -9 & (1) \\ 2x - 5y + z = 3 & (2) \\ -3x + 6y + 2z = 8 & (3) \end{cases}$$

C Infinite Number of Solutions (I)

(Line Intersection – Non Parallel Planes and Coplanar Normal Vectors)



Ex 2. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + y + 2z = -2 \\ 3x - y + 14z = 6 \\ x + 2y = -5 \end{cases}$$

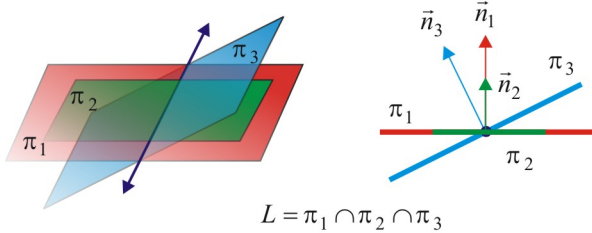
In this case:

- ⇒ The planes are *not parallel* but their normal vectors are *coplanar*: $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$.
- ⇒ The intersection is a *line*.
- ⇒ One scalar equation is a *combination* of the other two equations.
- ⇒ By solving the system (*), you may express two variables in terms of the third one using two equations.

D Infinite Number of Solutions (II)

(Line Intersection – Two Coincident Planes and one Intersecting Plane)

In this case:



- ⇒ Two planes are *coincident* and the third plane is *not parallel* to the coincident planes.
- ⇒ The intersection is a *line*.
- ⇒ Two scalar equations are *equivalent*. The coefficients A, B, C, D are *proportional* for these two equations.
- ⇒ You may express two variables in terms of the third one using two non equivalent equations.

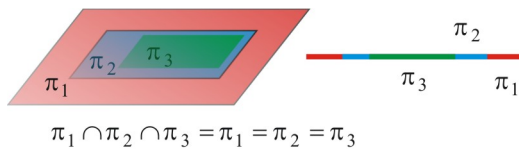
Ex 3. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + y - z = 2 & (1) \\ x - 2y + z = 4 & (2) \\ 2x - 4y + 2z = 8 & (3) \end{cases}$$

E Infinite Number of Solutions (III)

(Plane Intersection – Three Coincident Planes)

In this case:



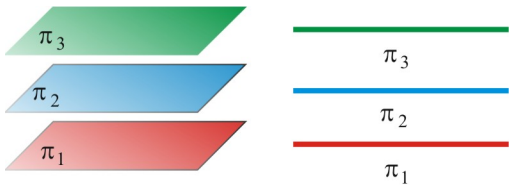
- ⇒ The coefficients A, B, C, D are *proportional* for all three equations.
- ⇒ Any point of one plane is also a point on the other two planes.
- ⇒ The intersection is a *plane*.

Ex 4. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x - y - 2z = 1 \\ 2x - 2y - 4z = 2 \\ -4x + 4y + 8z = -4 \end{cases}$$

**F No Solution
(Parallel and Distinct Planes)**

In this case:



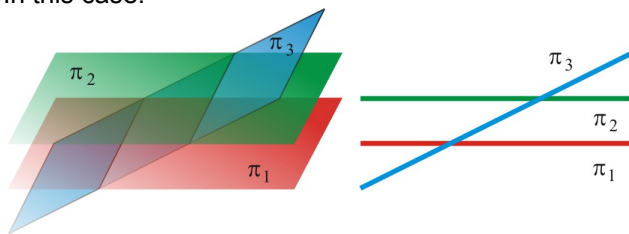
- ⇒ There are three *parallel* and *distinct* planes.
- ⇒ There is *no point of intersection*.
- ⇒ There is *no solution* for the system of equations (the system of equations is *incompatible*).
- ⇒ The coefficients A, B, C are *proportional* but the coefficients A, B, C, D are *not proportional*.
- ⇒ By solving the system (*) you get *false* statements (like $0 = 1$).

Ex 5. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + 2y + 3z = 1 & (1) \\ 2x + 4y + 6z = -1 & (2) \\ -x - 2y - 3z = 3 & (3) \end{cases}$$

**G No Solution (II)
(H Configuration)**

In this case:



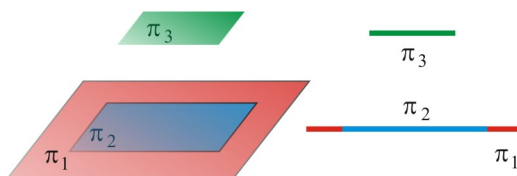
- ⇒ Two planes are *parallel and distinct* and the third plane is *intersecting*.
- ⇒ There is *no point of intersection*.
- ⇒ The coefficients A, B, C are *proportional* for two planes.
- ⇒ There is *no solution* for the system of equations (the system of equations is *incompatible*).
- ⇒ By solving the system (*) you get *false* statements (like $0 = 1$).

Ex 6. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + y - z = 1 & (1) \\ x + y + z = 2 & (2) \\ -2x - 2y + 2z = 3 & (3) \end{cases}$$

H No Solution (III)

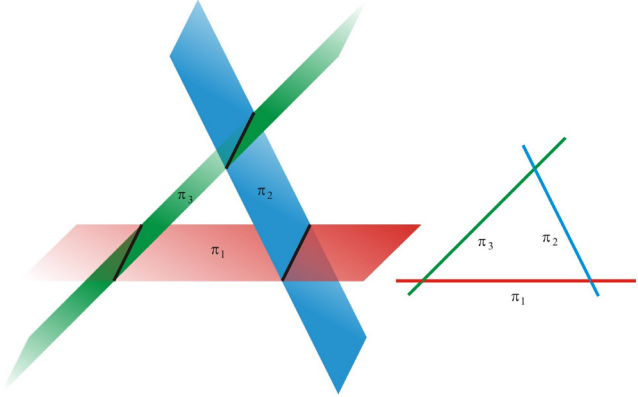
In this case:



- ⇒ Three planes are *parallel* but only *two are*

Ex 7. Solve the following system of equations. Give a geometric interpretation of the solution(s).

$$\begin{cases} x + 2y + 3z = 1 & (1) \\ 3x + 6y + 9z = 3 & (2) \\ -2x - 4y - 6z = 2 & (3) \end{cases}$$

<p><i>coincident.</i></p> <ul style="list-style-type: none"> ⇒ The coefficients A, B, C are <i>proportional</i> for all equations but the coefficients A, B, C, D are <i>proportional</i> only for two planes. ⇒ There is <i>no solution</i> for the system of equations (the system of equations is <i>incompatible</i>). ⇒ By solving the system (*) you get <i>false</i> statements (like $0 = 1$). 	
<p>I No Solution (IV) (Delta Configuration)</p> <p>In this case:</p>  <ul style="list-style-type: none"> ⇒ The planes are <i>not parallel</i> (the coefficients A, B, C are <i>not proportional</i>). ⇒ The normal vectors are <i>coplanar</i> ($\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$). ⇒ There is <i>no point of intersection</i> between all three planes. ⇒ There is <i>no solution</i> for the system of equations (the system of equations is <i>incompatible</i>). ⇒ By solving the system (*) you get <i>false</i> statements (like $0 = 1$). 	<p>Ex 8. Solve the following system of equations. Give a geometric interpretation of the solution(s).</p> $\begin{cases} 2x + y + z = 1 & (1) \\ -x + y + z = -1 & (2) \\ x + y + z = 0 & (3) \end{cases}$

Reading: Nelson Textbook, Pages 520-529

Homework: Nelson Textbook: Page 530 #8, 9, 10, 13