9.1 Intersection of two Lines

A Relative Position of two Lines
Two lines may be:

a) parallel and distinct

b) parallel and coincident

c) intersecting (not parallel, not intersecting)

d) skew

B Intersection of two Lines (Algebraic Method)
The point of intersection of two lines \( L_1 : \vec{r} = \vec{r}_{01} + t\vec{u}_1, \ t \in \mathbb{R} \)
and \( L_2 : \vec{r} = \vec{r}_{02} + s\vec{u}_2, \ s \in \mathbb{R} \) is given by the solution of the following system of equations (if exists):

\[
\begin{align*}
    x_{01} + tu_{x1} &= x_{02} + su_{x2} \\
    y_{01} + tu_{y1} &= y_{02} + su_{y2} \\
    z_{01} + tu_{z1} &= z_{02} + su_{z2}
\end{align*}
\]

(*)

Hint: Solve by substitution or elimination the system of two equations and check if the third is satisfied.

C Unique Solution
If by solving the system (*), you end by getting a unique value for \( t \) and \( s \) satisfying this system, then the lines have a unique point of intersection.

To get this point, substitute either the \( t \) value into the line \( L_1 \) equation or substitute the \( s \) value into the line \( L_2 \) equation.

D Infinite Number of Solutions
If by solving the system (*), you end by getting two true statements (like \( 2 = 2 \)) and one equation in \( s \) and \( t \), then there exist an infinite number of solutions of the system (*).

Therefore the lines intersect into an infinite number of points.

In this case the lines are parallel and coincident.

Ex 1. Find the point(s) of intersection of the following two lines. Show that this point is unique.

\( L_1 : \vec{r} = (0,1,2) + t(1,-1,2), \ t \in \mathbb{R} \)
\( L_2 : \vec{r} = (-3,4,-4) + s(0,1,2), \ s \in \mathbb{R} \)

Ex 2. Find the point(s) of intersection of the following two lines. Show that there are an infinite number of points of intersections and therefore the lines are parallel and coincident.

\( L_1 : \vec{r} = t(0,-1,2), \ t \in \mathbb{R} \)
\( L_2 : \vec{r} = (0,-6,12) + s(0,3,-6), \ s \in \mathbb{R} \)
### E No Solution (Parallel Lines)

If by solving the system (*) you get at least one *false* statement (like \(0 = 1\)) then the system has *no solution*. Therefore, the lines have *no point of intersection*. If, in addition, the lines are *parallel* (\(\vec{u}_1 \times \vec{u}_2 = \vec{0}\)), then the lines are *parallel and distinct*.

### F No Solutions (Skew Lines)

If by solving the system (*) you get at least one *false* statement (like \(0 = 1\)) then the system has *no solution*. Therefore, the lines have *no point of intersection*. If, in addition, the lines are not *parallel* (\(\vec{u}_1 \times \vec{u}_2 \neq \vec{0}\)), then the lines are *skew*.

### G Classifying Lines (Vector Method)

<table>
<thead>
<tr>
<th>Lines</th>
<th>Parallel Lines</th>
<th>Non Parallel Lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\vec{r}<em>{01} - \vec{r}</em>{02}) \times \vec{u}_1 = \vec{0})</td>
<td>((\vec{r}<em>{01} - \vec{r}</em>{02}) \times \vec{u}_1 \neq \vec{0})</td>
<td>((\vec{r}<em>{01} - \vec{r}</em>{02}) \cdot (\vec{u}_1 \times \vec{u}_2) = 0)</td>
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</tbody>
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<table>
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<tr>
<th>Parallel Coincident Lines</th>
<th>Parallel Distinct Lines</th>
<th>Non Parallel Intersecting Lines</th>
<th>Non Parallel Skew Lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\vec{u}_1 \times \vec{u}_2 = \vec{0})</td>
<td>(\vec{u}_1 \times \vec{u}_2 \neq \vec{0})</td>
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</tr>
</tbody>
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Ex 3. Find the point(s) of intersection of the following two lines. Show that there is no point of intersection and the lines are parallel and distinct.

L₁ : \(\vec{r} = (-2,3,1) + t(1,-2,1)\), \(t \in \mathbb{R}\)
L₂ : \(\vec{r} = (0,2,1) + s(-2,4,-2)\), \(s \in \mathbb{R}\)

Ex 4. Find the point(s) of intersection of the following two lines. Show that there is no point of intersection and the lines not parallel, therefore the lines are skew.

L₁ : \(\vec{r} = (1,-1,0) + t(0,0,1)\), \(t \in \mathbb{R}\)
L₂ : \(\vec{r} = (-2,1,0) + s(1,0,0)\), \(s \in \mathbb{R}\)
Ex 5. Use the vector method presented above to classify each pair of lines as parallel and distinct, parallel and coincident, not parallel and intersecting or not parallel and skew.

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<tbody>
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<td>a)</td>
<td>b)</td>
</tr>
<tr>
<td>( L_1 : \vec{r} = (0,1,2) + t(1,2,3), \quad t \in \mathbb{R} )</td>
<td>( L_1 : \vec{r} = t(1,−1,0), \quad t \in \mathbb{R} )</td>
</tr>
<tr>
<td>( L_2 : \vec{r} = (−2,−1,0) + s(−2,−4,−6), \quad s \in \mathbb{R} )</td>
<td>( L_2 : \vec{r} = (−4,4,0) + s(3,−3,0), \quad s \in \mathbb{R} )</td>
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<th>c)</th>
<th>d)</th>
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Ex 6. Prove that (if exists) the point of intersection between two lines \( L_1 : \vec{r} = \vec{r}_{01} + t\vec{u}_1, \quad t \in \mathbb{R} \) and \( L_2 : \vec{r} = \vec{r}_{02} + s\vec{u}_2, \quad s \in \mathbb{R} \) is given by the vector formula:

\[
\vec{r} = \vec{r}_{01} + \frac{[(\vec{r}_{02} - \vec{r}_{01}) \times \vec{u}_2] \cdot (\vec{u}_1 \times \vec{u}_2)}{\left\| \vec{u}_1 \times \vec{u}_2 \right\|^2} \vec{u}_1
\]

Reading: Nelson Textbook, Pages 489-496
Homework: Nelson Textbook: Page 497 #8, 9, 10, 11, 12, 13, 14, 15, 18