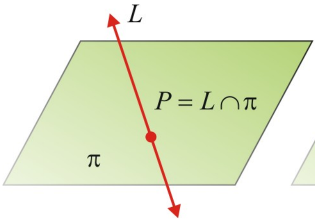
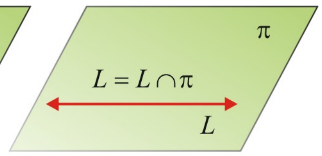
9.1 Intersection of a Line with a Plane

A Relative Position of a Line and a Plane

There are three possible situations as represented below:



a) The line *intersects* the plane at a single point.



b) The line *lies* on the plane. There are an infinite number of points of intersections.

$L \cap \pi = \varnothing$

c) The line is *parallel* to the plane but *distinct*. There is no point of intersection.

B Intersection of a Line and a Plane (Algebraic Method)

To get the intersection between a line L and a plane π:

a) *Substitute* the parametric equations of the line

$$L: \begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \\ z = z_0 + tu_z \end{cases} \quad (1)$$

into the Cartesian equation of the plane

$$\pi : Ax + By + Cz + D = 0 \quad (2)$$

to get the equation:

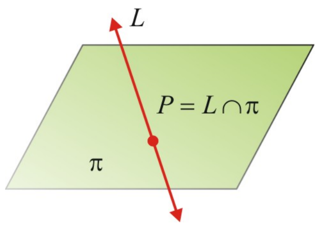
$$A(x_0 + tu_x) + B(y_0 + tu_y) + C(z_0 + tu_z) + D = 0 \quad (*)$$

b) *Solve* (if possible) the equation (*) for the parameter t.

c) *Substitute* the value of the parameter t into the parametric equations of the line (1) to get the point of intersection.

C Unique Solution (Point Intersection)

In this case, by solving the equation (*) you get a *unique value* for the parameter t. Therefore, there is a unique *point of intersection* between the line and the plane.



The line *intersects* the plane at a unique point.

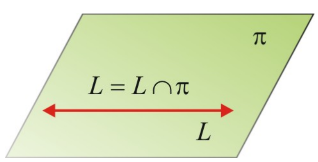
Ex 1. Find the point(s) of intersection between the line $L : \vec{r} = (-6,9,-1) + t(-2,3,1), \quad t \in R$ and the plane $\pi : -x + 2y + z + 4 = 0$.

D Infinite Number of Solutions (Line Intersection)

In this case, by solving the equation (*) you get the equation:

$$0t = 0$$

which has an *infinite number of solutions*. Therefore, there are an *infinite number of points of intersection*.



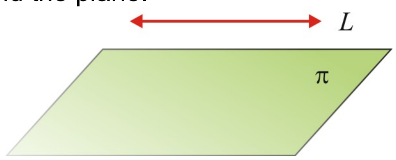
The line is *lies* on the plane.

Ex 2. Find the point(s) of intersection between the line $L : \vec{r} = (3,0,0) + t(0,2,-3), \quad t \in R$ and the plane $\pi : -2x + 3y + 2z + 6 = 0$.

E No Solution (No Intersection)
In this case, by solving the equation (*) you get a false statement like:

$$0t = 1$$

The equation *does not have any solution* and therefore there is *no point of intersection* between the line and the plane.

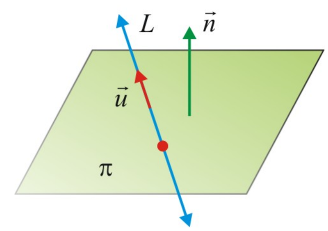


The line is *parallel* to the plane and *does not lie* on the plane.

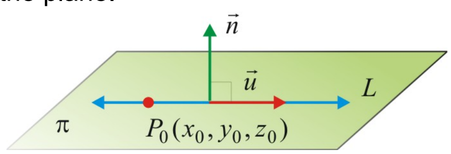
Ex 3. Ex 2. Find the point(s) of intersection between the line $L: \vec{r} = (1,2,3) + t(0,1,1), \quad t \in R$ and the plane $\pi: x + y - z - 3 = 0$.

F Classifying Lines
Let consider the line $L: \vec{r} = \vec{r}_0 + t\vec{u}, \quad t \in R$, where $P_0(x_0, y_0, z_0)$ is a specific point on the line, and the plane $\pi: Ax + By + Cz + D = 0$, where $\vec{n} = (A, B, C)$ is a normal vector to the plane.

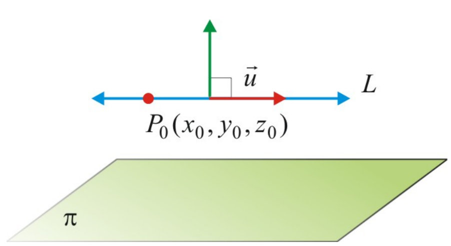
a) If $\vec{n} \cdot \vec{u} \neq 0$ the line *intersects* the plane at a unique point.



b) If $\vec{n} \cdot \vec{u} = 0$ and $Ax_0 + By_0 + Cz_0 + D = 0$ then the line *lies* on the plane.



c) If $\vec{n} \cdot \vec{u} = 0$ and $Ax_0 + By_0 + Cz_0 + D \neq 0$ then the line is *parallel* to the plane but *does not lie* on the plane.



Note. By solving the equation (*) for t you will end by getting the same cases and conditions as above. Try this as an exercise.

Ex 4. Consider the plane $\pi: 4x + 3y - 2z + 12 = 0$. Classify each line as intersecting the plane, contained by the plane, or distinct from the plane. Do not find the point(s) of intersection using the algebraic method.

a) $L: \vec{r} = (-3,0,0) + t(0,2,3), \quad t \in R$

b) $L: \vec{r} = (1,0,-2) + t(1,-2,0), \quad t \in R$

c) $L: \vec{r} = t(1,0,2), \quad t \in R$

Ex 5. Show that the point P of intersection between the plane $\pi: \vec{r} = \vec{p}_0 + s\vec{u} + t\vec{v}$ and the line $L: \vec{r} = \vec{l}_0 + q\vec{w}$ is given by: $\vec{r}_P = \vec{l}_0 + \dfrac{(\vec{p}_0 - \vec{l}_0) \cdot (\vec{u} \times \vec{v})}{\vec{w} \cdot (\vec{u} \times \vec{v})} \vec{w}$. Explain.

Reading: Nelson Textbook, Pages 489-496
Homework: Nelson Textbook: Page 497 #1, 5, 7, 17

