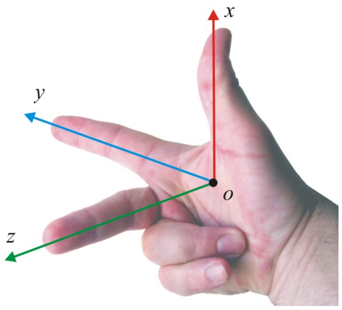
7.6 Cross Product

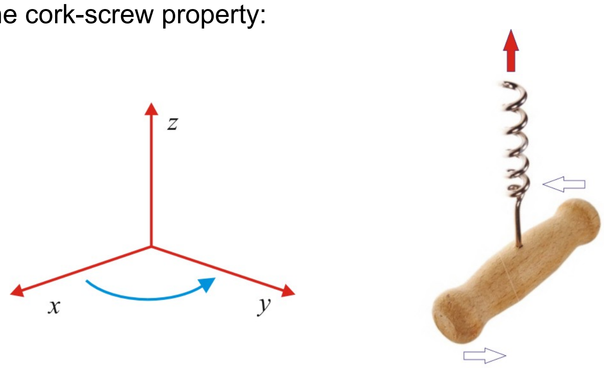
A Right Hand System

The *Right Hand System* is based on the position of first three fingers of the right hand as illustrated on the following figure:



B Cork-Screw Rule

The *cork-screw rule* describes a *right hand system* based on the cork-screw property:



If you rotate the x-axis towards the y-axis using the shortest path, the screw goes in the positive direction of the z-axis.

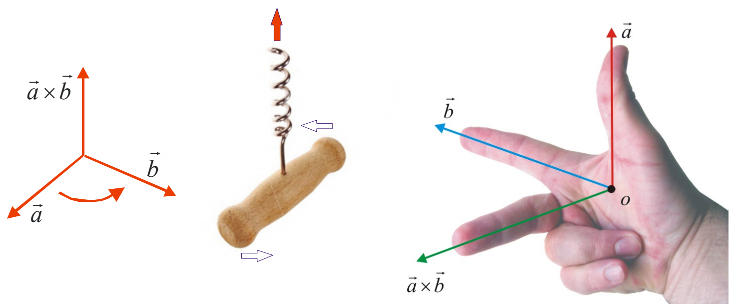
C Cross Product

The *cross product* between two vectors $\vec{a}$ and $\vec{b}$ is a *vector* quantity denoted by $\vec{a}\times\vec{b}$ having the following properties:

a) $\|\vec{a}\times\vec{b}\|=\|\vec{a}\|\|\vec{b}\|\sin\alpha$ where $\alpha=\angle(\vec{a},\vec{b})$

b) $\vec{a}\times\vec{b}$ is *perpendicular to both* $\vec{a}$ and $\vec{b}$ (is perpendicular to the plane determined by $\vec{a}$ and $\vec{b}$)

c) the vectors $\vec{a}$, $\vec{b}$, and $\vec{a}\times\vec{b}$ form a *right-handed system*



D Specific Cases

1. If $\vec{a}\parallel\vec{b}$ ($\alpha=0$ or $\alpha=\pi=180°$), then $\vec{a}\times\vec{b}=\vec{0}$.
2. If $\vec{a}\perp\vec{b}$ ($\alpha=\pi/2=90°$), then $\|\vec{a}\times\vec{b}\|=\|\vec{a}\|\|\vec{b}\|=maximum$
3. If $\vec{a}\equiv\vec{b}$ then $\vec{a}\times\vec{a}=\vec{0}$.

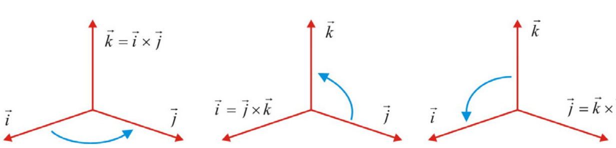
Ex 1. The magnitudes of two vectors $\vec{a}$ and $\vec{b}$ are $\|\vec{a}\|=2$ and $\|\vec{b}\|=3$ respectively, and the angle between them is $\alpha=60°$. Find the magnitude of the cross product of these vectors.

E Cross Product of Unit Vectors

The cross product of the *standard unit vectors* is given by:

$$\vec{i}\times\vec{i}=\vec{0}\qquad \vec{j}\times\vec{j}=\vec{0}\qquad \vec{k}\times\vec{k}=\vec{0}$$

$$\vec{i}\times\vec{j}=\vec{k}\qquad \vec{j}\times\vec{k}=\vec{i}\qquad \vec{k}\times\vec{i}=\vec{j}$$



D Cross Product of two Algebraic Vectors

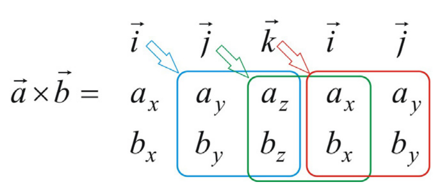
The cross product of two algebraic vectors

$\vec{a}=(a_x,a_y,a_z)=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$ and

$\vec{b}=(b_x,b_y,b_z)=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$ is given by:

$$\vec{a}\times\vec{b}=\vec{i}(a_yb_z-a_zb_y)+\vec{j}(a_zb_x-a_xb_z)+\vec{k}(a_xb_y-a_yb_x)$$

$$=\vec{i}\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}+\vec{j}\begin{vmatrix}a_z & a_x\\ b_z & b_x\end{vmatrix}+\vec{k}\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$



Ex 2. For each case, find the cross product of the vectors $\vec{a}$ and $\vec{b}$.

a) $\vec{a} = (1,-2,0)$, $\vec{b} = (0,-1,2)$

b) $\vec{a} = -\vec{i} + 2\vec{j}$, $\vec{b} = \vec{i} - 2\vec{j} - \vec{k}$

c) $\vec{a} = (-1,1,-2)$, $\vec{b} = -2\vec{i} - \vec{j} + 3\vec{k}$

E Properties of Cross Product

The following properties apply for the cross product:

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (anti-commutative property)
2. $\lambda(\vec{a} \times \vec{b}) = (\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (distributive property)
4. $\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a} = \vec{0}\ or\ \vec{b} = \vec{0}\ or\ \vec{a} \parallel \vec{b}$
5. $\vec{a} \times \vec{0} = \vec{0}$
6. $\vec{a} \times \vec{a} = \vec{0}$

Note: The dot and cross products have a higher priority in comparison to addition and subtraction operations.

d) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{b} \cdot \vec{a})\vec{c}$ (triple cross product)

Ex 3. Use the cross product properties to prove the following relations:

a) $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$

b) $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) + (\vec{a} \cdot \vec{b})(\vec{a} \cdot \vec{b}) = (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b})$

c) $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$ (mixed product)

See part d) on the left side.

Ex 4. Find an unit vector perpendicular to both $\vec{a} = (0,1,1)$ and $\vec{b} = (1,1,0)$.

Ex 5. Classify as scalar, vector, or meaningless.

a) $\vec{a} + \vec{b} \times \vec{c}$
b) $\vec{a} + \vec{b} \cdot \vec{c}$
c) $\vec{a} \times \vec{b} - \vec{b} \times \vec{c}$
d) $(\vec{b} \cdot \vec{c})\vec{a}$
e) $(\vec{b} \cdot \vec{c}) \times \vec{a}$
f) $(\vec{b} \cdot \vec{c})(\vec{a} \times \vec{b})$
g) $(\vec{b} - \vec{c})(\vec{a} \times \vec{b})$
h) $(\vec{b} \cdot \vec{c})(\vec{a} \times \vec{b}) - (\vec{b} + \vec{c}) \times (\vec{c} \times \vec{b})$
i) $(\vec{b} \cdot \vec{c}) - \vec{a} \cdot (\vec{b} \times \vec{c})$

Reading: Nelson Textbook, Pages 401-407
Homework: Nelson Textbook: Page 407 #3, 4ab, 5, 8a, 11, 13

