6.7 Operations with Algebraic Vectors in R^3

A 3D Algebraic Vectors
A 3D Algebraic Vector may be written in components form as:

$$\vec{v} = (v_x, v_y, v_z)$$

or in terms of unit vectors as:

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

and has a magnitude given by:

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$

Ex 1. Consider the vector $\vec{a} = -\vec{i} + 3\vec{j} - 2\vec{k}$.

a) Write the vector in components form.

b) Find the magnitude of the vector $\vec{a}$.

B Addition of 3D Algebraic Vectors
The sum of two 3D algebraic vectors $\vec{a} = (a_x, a_y, a_z) = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ and $\vec{b} = (b_x, b_y, b_z) = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$ is a 3D algebraic vector given by:

$$\vec{a} + \vec{b} = (a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) + (b_x\vec{i} + b_y\vec{j} + b_z\vec{k})$$
$$= (a_x + b_x)\vec{i} + (a_y + b_y)\vec{j} + (a_z + b_z)\vec{k}$$
$$\vec{a} + \vec{b} = (a_x, a_y, a_z) + (b_x, b_y, b_z)$$
$$= (a_x + b_x, a_y + b_y, a_z + b_z)$$

Ex 2. Find the sum of the vector $\vec{a} = -2\vec{i} + 5\vec{j} - \vec{k}$ and $\vec{b} = (2, 0, -3)$.

C Subtraction of 3D Algebraic Vectors
The difference of two 3D algebraic vectors $\vec{a} = (a_x, a_y, a_z) = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ and $\vec{b} = (b_x, b_y, b_z) = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$ is a 3D algebraic vector given by:

$$\vec{a} - \vec{b} = (a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) - (b_x\vec{i} + b_y\vec{j} + b_z\vec{k})$$
$$= (a_x - b_x)\vec{i} + (a_y - b_y)\vec{j} + (a_z - b_z)\vec{k}$$
$$\vec{a} - \vec{b} = (a_x, a_y, a_z) - (b_x, b_y, b_z)$$
$$= (a_x - b_x, a_y - b_y, a_z - b_z)$$

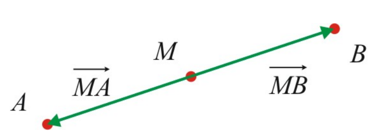
Ex 3. Find the magnitude of the difference $\vec{a} - \vec{b}$ between the vector $\vec{a} = \vec{i} - \vec{j}$ and $\vec{b} = (1, 2, -1)$.

D Multiplication of 3D Algebraic Vector by a Scalar
The multiplication of a 3D algebraic vector $\vec{a} = (a_x, a_y, a_z) = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ by a scalar λ is a 3D algebraic vector given by:

$$\lambda\vec{a} = \lambda(a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) = (\lambda a_x)\vec{i} + (\lambda a_y)\vec{j} + (\lambda a_z)\vec{k}$$
$$\lambda\vec{a} = \lambda(a_x, a_y, a_z) = (\lambda a_x, \lambda a_y, \lambda a_z)$$

Ex 4. Given $\vec{a} = (1, -2, 0)$, $\vec{b} = (0, -2, -3)$, and $\vec{c} = (-1, 0, 2)$, find the vector $\vec{d} = \vec{a} - 2\vec{b} + 3\vec{c}$.

E Midpoint
The midpoint of the segment line AB is the point M such that $\overrightarrow{MA} + \overrightarrow{MB} = \vec{0}$.



Ex 5. Prove that the midpoint of the segment line AB is given by:

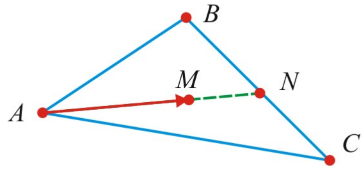
$$\overrightarrow{OM} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2} = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}, \frac{z_A + z_B}{2}\right)$$

F Centroid

The centroid of a system of points P_1, P_2, …, P_n is the point C defined by:

$$\overrightarrow{OC} = \frac{\overrightarrow{OP_1} + \overrightarrow{OP_2} + \ldots + \overrightarrow{OP_n}}{n}$$

Ex 6. Consider the triangle ΔABC where $A(-1,-4,1)$, $B(2,-3,0)$, and $C(-4,1,2)$.



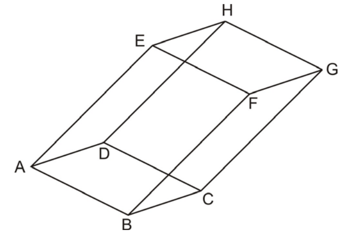
a) Find the centroid M of the triangle.

b) Use the result at part a) to show that $\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} = \vec{0}$.

c) Find the midpoint N of the side BC.

d) Show that $\overrightarrow{AN} = 3\overrightarrow{MN}$.

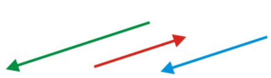
Ex 7. The shape $ABCDEFGH$ is a parallelepiped. Given $A(0,1,3)$, $B(1,0,2)$, $C(1,2,0)$, and $E(4,4,4)$, find the coordinates of all the other vertices. See the figure below.



G Parallelism

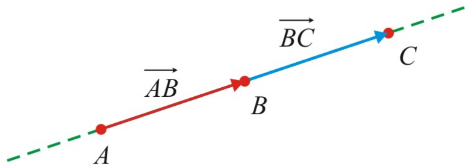
Two vectors $\vec{a}$ and $\vec{b}$ are parallel ($\vec{a} \parallel \vec{b}$) if there exists λ such that $\vec{a} = \lambda \vec{b}$.

Note that parallel vectors may have same direction or opposite direction:



Ex 8. Prove that the vectors $\vec{a} = (2,4,-6)$ and $b(-1,-2,3)$ are parallel.

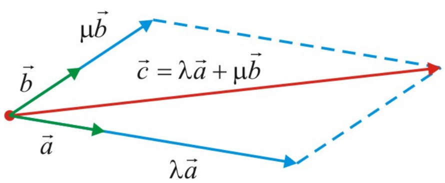
H Co-linearity
Three points A, B, and C are collinear if $\overrightarrow{AB} \parallel \overrightarrow{BC}$.



Ex 9. Prove that the points $A(2,-1,0)$, $B(-1,0,2)$, and $C(0,1,2)$ are not collinear.

I Linear Dependency
Three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linear dependent if there exist λ and μ such that $\vec{c} = \lambda\vec{a} + \mu\vec{b}$.

Note. In order to be linear dependant the vectors must be coplanar (must belong to the same plan).



Ex 10. Prove that the vectors $\vec{a} = (-1,2,-3)$, $\vec{b} = (2,0,-1)$, and $\vec{c} = (-7,6,-7)$ are linear dependant.

Reading: Nelson Textbook, Pages 327-332
Homework: Nelson Textbook: Page 332 #1, 3, 5b, 7c, 11, 12, 15