4.3 Vertical and Horizontal Asymptotes

**A Vertical Asymptote**
If the value of \( f(x) \) can be made arbitrarily large by taking \( x \) sufficiently close to \( a \) with \( x < a \) then:
\[
\lim_{x \to a^-} f(x) = \infty
\]
The line \( x = a \) is called vertical asymptote to the graph of \( y = f(x) \).

**B Notes**
In this case, writing \( \lim_{x \to a^-} f(x) = \infty \) is better than writing \( \lim_{x \to a^-} f(x) = \) DNE.

**C Rational Functions**
A rational function of the form \( f(x) = \frac{p(x)}{q(x)} \) has a vertical asymptote \( x = a \) if:
\[
q(a) = 0 \text{ and } p(a) \neq 0
\]

**Ex 1.** Find the equation of the vertical asymptote(s) for \( y = f(x) = \frac{x - 2}{x^2 - 4} \).

**Ex 2.** Find the behavior of the function \( y = f(x) = \frac{-x}{x^2 + 2x - 3} \) near the vertical asymptotes.
**D Horizontal Asymptote**

A horizontal line $y = L$ is called horizontal asymptote to the graph of $y = f(x)$ if:

$$\lim_{x \to \pm\infty} f(x) = L \quad \text{or} \quad \lim_{x \to \pm\infty} f(x) = L$$

Notes.
1. A horizontal asymptote may be crossed or touched by the graph of the function.
2. The graph of a function may have at most two horizontal asymptotes (one as $x \to -\infty$ and one as $x \to \infty$) (see the figure on the right).

**E Limits to Infinity**

If $n \geq 1$, then:

$$\lim_{x \to \pm\infty} x^n = (\pm\infty)^n$$

$$\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$$

**F End behaviour**

A polynomial functions $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ behaves at infinity as the leading term $a_n x^n$.

**G Rational Functions**

To compute limits at infinity for a rational function, use the end behavior of the numerator and denominator:

$$\lim_{x \to \pm\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_2 x^2 + b_1 x + b_0} = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}$$

**Ex 3.** Find the equation of the horizontal asymptote(s) to the graph of the function $y = f(x)$ represented graphically below.

**Ex 4.** Compute each limit.

a) $\lim_{x \to \infty} x^n$

b) $\lim_{x \to \infty} x^2$

c) $\lim_{x \to \infty} x^3$

d) $\lim_{x \to \infty} x^4$

**Ex 5.** Compute each limit.

a) $\lim_{x \to \infty} (-3x^4 + 5x^3 - 4)$

b) $\lim_{x \to \infty} (-x^3 - 2x^2 + x)$

**Ex 6.** Compute each limit.

a) $\lim_{x \to \infty} \frac{-2x^3 + x}{5x^3 + x^2 + 1}$

b) $\lim_{x \to \infty} \frac{3x^2 + 1}{-2x^3 + 3x}$

c) $\lim_{x \to \infty} \frac{3x^4 + x^2 - x}{-x^2 + x - 1}$
**H Horizontal Asymptotes for Rational Functions**

A rational function of the form:

\[ f(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0}{b_mx^m + b_{m-1}x^{m-1} + \ldots + b_1x + b_0} \]

has
- a horizontal asymptote \( y = 0 \) if \( m > n \)
- a horizontal asymptote \( y = \frac{a_n}{b_m} \) if \( m = n \)
- no horizontal asymptote if \( n > m \)

**Notes:**
- A rational function may have at most one horizontal asymptote.

**Ex 7. Find the equation of the horizontal asymptote.**

a) \( f(x) = \frac{3x^4 + 1}{-2x^4 + 3x^2} \)

b) \( f(x) = \frac{-x^2 + 2x}{x^3 - x^2 + 2} \)

c) \( f(x) = \frac{2x^2 - 3}{x+1} \)

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**I Oblique (Slant) Asymptote**

The line \( y = ax + b \) is an oblique (slant) asymptote for the curve \( y = f(x) \) if:

\[ \lim_{x \to \pm\infty} [f(x) - (ax + b)] = 0 \]

**Notes:**
1. An oblique asymptote may be crossed or touched by the graph of the function.
2. The graph of a function may have at most two oblique asymptotes (one as \( x \to -\infty \) and one as \( x \to \infty \)).

**Ex 8. Find the equations of the oblique asymptotes for the function represented below (oblique asymptotes are also represented in the figure).**

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**J Oblique Asymptotes for Rational Functions**

A rational function of the form:

\[ f(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0}{b_mx^m + b_{m-1}x^{m-1} + \ldots + b_1x + b_0} \]

has an oblique (slant) asymptote if \( n = m+1 \).

**Note.** To get the equation of the oblique (slant) asymptote, use the **long division algorithm** to write the rational function in the form:

\[ f(x) = \frac{P_n(x)}{Q_m(x)} = ax + b + \frac{R(x)}{Q_m(x)} \]

where \( 0 \leq \text{degree}(R) < \text{degree}(Q_m) = m \)

Finally, the equation of the oblique (slant) asymptote is given by:

\[ y = ax + b \]

**Ex 9. Consider the rational function** \( y = f(x) = \frac{x^2}{x-1} \).

a) Find the equation of the oblique asymptote.

b) Find the derivative function and simplify.

c) Find the local extrema.