4.3 Vertical and Horizontal Asymptotes

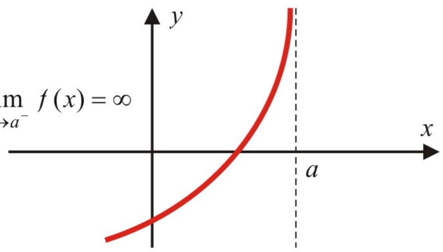
A Vertical Asymptote

If the value of $f(x)$ can be made *arbitrarily large* by taking x *sufficiently close* to a with $x < a$ then:

$$\lim_{x \to a^-} f(x) = \infty$$

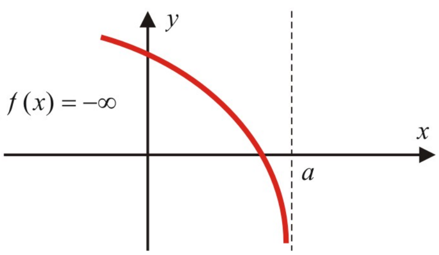
The line $x = a$ is called *vertical asymptote* to the graph of $y = f(x)$.

$\lim_{x \to a^-} f(x) = \infty$



Similarly:

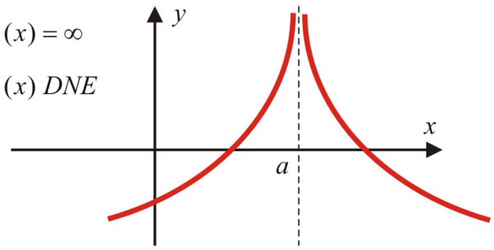
$\lim_{x \to a^-} f(x) = -\infty$



B Notes

$\lim_{x \to a} f(x) = \infty$

$\lim_{x \to a} f(x)\ DNE$

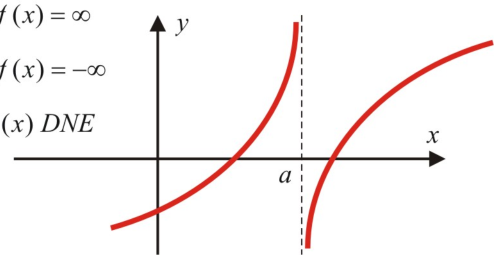


In this case, writing $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = \infty$ is better than writing $\lim_{x \to a} f(x)\ DNE$.

$\lim_{x \to a^-} f(x) = \infty$

$\lim_{x \to a^+} f(x) = -\infty$

$\lim_{x \to a} f(x)\ DNE$



In this case, $\lim_{x \to a} f(x)\ DNE$.

C Rational Functions

A rational function of the form $f(x) = \dfrac{p(x)}{q(x)}$ has a vertical asymptote $x = a$ if:

$q(a) = 0\ and\ p(a) \neq 0$

Ex 1. Find the equation of the vertical asymptote(s) for $y = f(x) = \dfrac{x-2}{x^2-4}$.

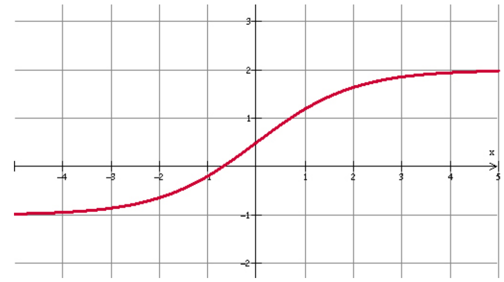
Ex 2. Find the behavior of the function $y = f(x) = \dfrac{-x}{x^2+2x-3}$ near the vertical asymptotes.

D Horizontal Asymptote
A horizontal line $y = L$ is called *horizontal asymptote* to the graph of $y = f(x)$ if:

$$\lim_{x\to\infty} f(x) = L \quad or \quad \lim_{x\to-\infty} f(x) = L$$

Notes.
1. A horizontal asymptote may be crossed or touched by the graph of the function.
2. The graph of a function may have *at most two* horizontal asymptotes (one as $x \to -\infty$ and one as $x \to \infty$) (see the figure on the right).

Ex 3. Find the equation of the horizontal asymptote(s) to the graph of the function $y = f(x)$ represented graphically below.



E Limits to Infinity
If $n \ge 1$, then:

$$\lim_{x\to\pm\infty} x^n = (\pm\infty)^n$$

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

Ex 4. Compute each limit.

a) $\lim_{x\to\infty} x$ b) $\lim_{x\to\infty} x^2$

c $\lim_{x\to-\infty} x^3$ d) $\lim_{x\to-\infty} x^4$

F End behaviour
A polynomial functions
$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$
behaves at infinity as the *leading term* $a_n x^n$.

Ex 5. Compute each limit.

a) $\lim_{x\to\infty}(-3x^4 + 5x^3 - 4)$

b) $\lim_{x\to-\infty}(-x^3 - 2x^2 + x)$

G Rational Functions
To compute *limits at infinity* for a rational function, use the *end behavior* of the numerator and denominator:

$$\lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_2 x^2 + b_1 x + b_0} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}$$

Ex 6. Compute each limit.

a) $\lim_{x\to\infty} \frac{-2x^3 + x}{5x^3 + x^2 + 1}$

b) $\lim_{x\to-\infty} \frac{3x^2 + 1}{-2x^3 + 3x}$

c) $\lim_{x\to-\infty} \frac{3x^4 + x^2 - x}{-x^2 + x - 1}$

H Horizontal Asymptotes for Rational Functions
A *rational function* of the form:

$$f(x)=\frac{P_n(x)}{Q_m(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+...+b_2x^2+b_1x+b_0}$$

has

- a horizontal asymptote $y=0$ if $m>n$
- a horizontal asymptote $y=\frac{a_n}{b_m}$ if $m=n$
- no horizontal asymptote if $n>m$

Note: A rational function may have *at most one* horizontal asymptote.

Ex 7. Find the equation of the horizontal asymptote.

a) $f(x)=\frac{3x^4+1}{-2x^4+3x^2}$

b) $f(x)=\frac{-x^2+2x}{x^3-x^2+2}$

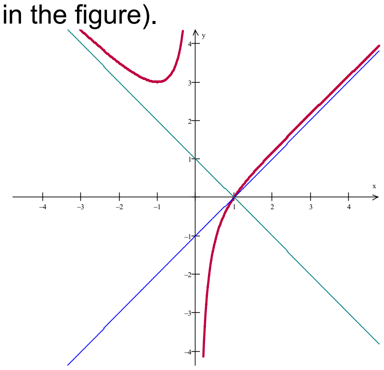
c) $f(x)=\frac{2x^2-3}{x+1}$

I Oblique (Slant) Asymptote
The line $y=ax+b$ is an *oblique (slant) asymptote* for the curve $y=f(x)$ if:

$$\lim_{x\to\pm\infty}[f(x)-(ax+b)]=0$$

Notes:
1. An oblique asymptote may be crossed or touched by the graph of the function.
2. The graph of a function may have at most two oblique asymptotes (one as $x\to-\infty$ and one as $x\to\infty$).

Ex 8. Find the equations of the oblique asymptotes for the function represented below (oblique asymptotes are also represented in the figure).



J Oblique Asymptotes for Rational Functions
A rational function of the form:

$$f(x)=\frac{P_n(x)}{Q_m(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+...+b_2x^2+b_1x+b_0}$$

has an *oblique (slant) asymptote* if $n=m+1$.

Note. To get the equation of the oblique (slant) asymptote, use the *long division algorithm* to write the rational function in the form:

$$f(x)=\frac{P_n(x)}{Q_m(x)}=ax+b+\frac{R(x)}{Q_m(x)}$$

where $0\le degree(R)<degree(Q_m)=m$

Finally, the equation of the oblique (slant) asymptote is given by:

$$y=ax+b$$

Ex 9. Consider the rational function $y=f(x)=\frac{x^2}{x-1}$.

a) Find the equation of the oblique asymptote.

b) Find the derivative function and simplify.

c) Find the local extrema.

Reading: Nelson Textbook, Pages 181-192
Homework: Nelson Textbook: Page 193 #1ab, 3bd, 4de, 5cd, 7ac, 9bd, 14

