2.5 Chain Rule

A Composition of functions

If $u = g(x)$ and $v = f(u)$ then:

$$x \underset{u=g(x)}{\rightarrow} u \underset{v=f(u)}{\rightarrow} v$$

and

$$v = f(u) = f(g(x)) = (f \circ g)(x)$$

B Chain Rule (Leibniz Notation)

$$\Delta x \underset{u=g(x)}{\rightarrow} \Delta u \underset{v=f(u)}{\rightarrow} \Delta v$$

and

$$\frac{\Delta v}{\Delta x} = \frac{\Delta v}{\Delta u}\frac{\Delta u}{\Delta x} \quad \rightarrow \quad \frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx}$$

Therefore:

$$\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx}$$

Ex 1. Consider $u = x^2 - x$ and $v = \sqrt{u}$. Find $\frac{dv}{dx}$.

Ex 2. Consider $u = \sqrt{x}$ and $v = \frac{u}{u-1}$. Find $\left.\frac{dv}{dx}\right|_{x=4}$.

C Composition of three functions

$$x \underset{u=h(x)}{\rightarrow} u \underset{v=g(u)}{\rightarrow} v \underset{w=f(v)}{\rightarrow} w$$

$$\frac{dw}{dx} = \frac{dw}{dv}\frac{dv}{du}\frac{du}{dx}$$

Ex 3. Consider $u = x^2$, $v = \frac{1}{u+1}$, and $w = \sqrt{v}$. Find $\frac{dw}{dx}$.

D Chain Rule (Lagrange Notation)

$$v = f(u) = f(g(x)) = (f \circ g)(x)$$
$$\frac{dv}{dx} \to [f(g(x))]'$$
$$\frac{dv}{du} \to f'(u) = f'(g(x)$$
$$\frac{du}{dx} \to g'(x)$$
$$\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx} \quad \to \quad [f(g(x))]' = f'(g(x))g'(x)$$

If g is differentiable at x and f is differentiable at $f(x)$ then the composition $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$(f \circ g)'(x) = [f(g(x))]' = f'(g(x))g'(x)$$

So, the derivative of $f(g(x))$ is the derivative of the *outside* function f evaluated of the *inside* function g times the derivative of the inside function g.

Ex 4. Differentiate $f(x) = (x^3 - 2x^2 + x)^5$.

Ex 5. Differentiate $f(x) = \sqrt[3]{x^2 - \sqrt{x}}$.

Ex 6. Differentiate $f(x) = x^2 \sqrt{\frac{x+1}{x^2+1}}$.

Ex 7. Differentiate $f(x) = x + \sqrt{x^2 + \sqrt{x^3 + \sqrt{x^4 + 1}}}$.

Reading: Nelson Textbook, Pages 94-95
Homework: Nelson Textbook: Page 95 #4f, 5b, 8, 9a, 12, 14, 15, 16