## 2.5 Chain Rule

### A Composition of functions

If $u = g(x)$ and $v = f(u)$ then:

$$ x \rightarrow u \rightarrow v \quad \text{and} \quad v = f(u) = f(g(x)) = (f \circ g)(x) $$

### B Chain Rule (Leibniz Notation)

$$ \frac{\Delta v}{\Delta x} = \frac{\Delta v}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \quad \Rightarrow \quad \frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx} $$

Therefore:

$$ \frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx} $$

### Ex 1. Consider $u = x^2 - x$ and $v = \sqrt{u}$. Find $\frac{dv}{dx}$.

### Ex 2. Consider $u = \sqrt{x}$ and $v = \frac{u}{u - 1}$. Find $\frac{dv}{dx} |_{x=4}$.

### C Composition of three functions

$$ x \rightarrow u \rightarrow v \rightarrow w \quad \text{and} \quad w = f(v) $$

$$ \frac{dw}{dx} = \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} $$

### Ex 3. Consider $u = x^2$, $v = \frac{1}{u + 1}$, and $w = \sqrt{v}$. Find $\frac{dw}{dx}$.
**D Chain Rule (Lagrange Notation)**

\[ v = f(u) = f(g(x)) = (f \circ g)(x) \]

\[ \frac{dv}{dx} \rightarrow [f(g(x))]' \]

\[ \frac{dv}{du} \rightarrow f'(u) = f'(g(x)) \]

\[ \frac{du}{dx} \rightarrow g'(x) \]

\[ \frac{dv}{dx} = \frac{dv}{du} \frac{du}{dx} \rightarrow [f(g(x))]' = f'(g(x))g'(x) \]

If \( g \) is differentiable at \( x \) and \( f \) is differentiable at \( f(x) \) then the composition \((f \circ g)(x) = f(g(x))\) is differentiable at \( x \) and

\[ (f \circ g)'(x) = [f(g(x))]' = f'(g(x))g'(x) \]

So, the derivative of \( f(g(x)) \) is the derivative of the *outside* function \( f \) evaluated at the *inside* function \( g \) times the derivative of the inside function \( g \).

| Ex 4. Differentiate | \( f(x) = (x^3 - 2x^2 + x)^5 \). |
| Ex 5. Differentiate | \( f(x) = \sqrt[3]{x^2 - \sqrt{x}} \). |
| Ex 6. Differentiate | \( f(x) = x^2 \sqrt{\frac{x+1}{x^2+1}} \). |
| Ex 7. Differentiate | \( f(x) = x + \sqrt{x^2 + \sqrt{x^3 + \sqrt{x^4 + 1}}} \). |

**Reading:** Nelson Textbook, Pages 94-95

**Homework:** Nelson Textbook: Page 95 #4f, 5b, 8, 9a, 12, 14, 15, 16