2.1 Derivative Function

A Derivative Function
Given a function $y = f(x)$, the *derivative function* of f is a *new function* called f' (*f prime*), defined at x by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

B Differentiability
A function $y = f(x)$ is called *differentiable* at x if $f'(x)$ exists.
A function $y = f(x)$ is differentiable over an open interval (a,b) if the function is differentiable at every number in that interval.
Note: The domain of derivative function f' is a subset of the domain of the original function f: $D_{f'} \subset D_f$. So a function is defined over D_f but is differentiable over $D_{f'}$.

C Interpretations of Derivative Function
1. The *slope of the tangent line* to the graph of $y = f(x)$ at the point $P(a, f(a))$ is given by $m = f'(a)$.
2. The *instantaneous rate of change* in the variable y with respect to the variable x, where $y = f(x)$, at $x = a$ is given by: $IRC = f'(a)$.

D Notations and Reading
$y' = f'(x)$ [Lagrange Notation]
Read: "y prime" or "f prime at x"

$\frac{dy}{dx} = \frac{d}{dx} f(x)$ [Leibnitz Notation]
Read: "dee y by dee x"

$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$
Read: "f prime at a, dee y by dee x at x equals a"

E First Principles
Differentiation is the process to find the derivative function for a given function.

First Principles is the process of differentiation by computing the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Ex 1. Use first principles to differentiate each function.
a) $f(x) = 2x - x^3$

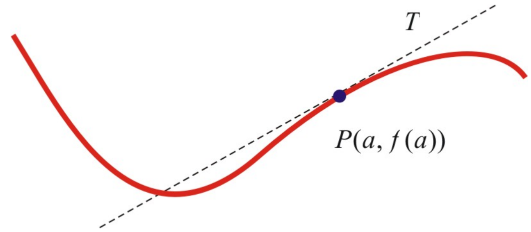
b) $f(x) = \frac{-3}{x^2}$

c) $f(x) = \sqrt{ax + b}$

G Non-Differentiability
A function is *not differentiable* at $x = a$ if $f'(a)$ *does not exist*.

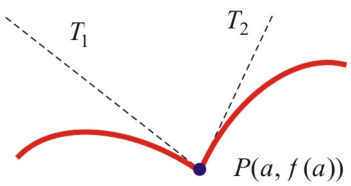
Notes:
1. If a function f is *not continuous* at $x = a$ then the function f is *not differentiable* at $x = a$.
2. If a function f is *continuous* at $x = a$ then the function f *may be or not* differentiable at $x = a$.

H Differentiability Point
If the function $y = f(x)$ is *differentiable* at $x = a$ then the tangent line at $P(a, f(a))$ is *unique* and *not vertical* (the slope of the tangent line is not ∞ or $-\infty$).



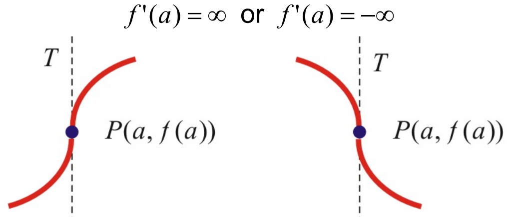
I Corner Point
$P(a, f(a))$ is a *corner point* if there are *two* distinct tangent lines at P, one for the left-hand branch and one for the right-hand branch. For example:

$$f(x) = \begin{cases} f_1(x), & x < a \\ f_2(x), & x > a \end{cases} \quad \text{and} \quad f_1'(a) \neq f_2'(a)$$



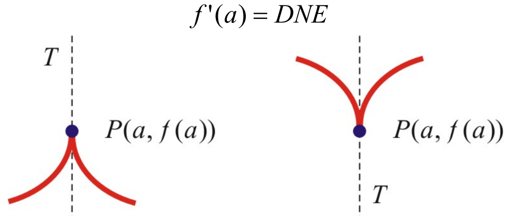
J Infinite Slope Point
$P(a, f(a))$ is a *infinite slope point* if the tangent line at P is vertical and the function is increasing or decreasing in the neighborhood at the of the point P.

$$f'(a) = \infty \text{ or } f'(a) = -\infty$$

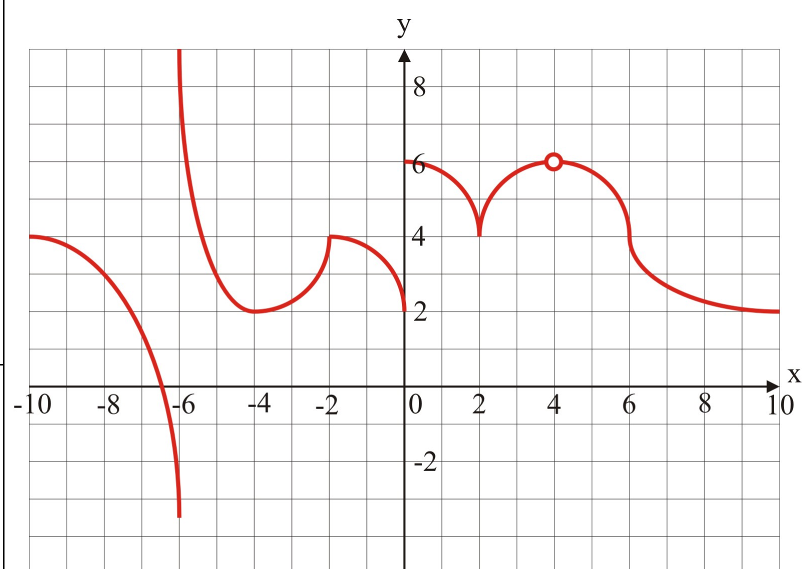


K Cusp Point
$P(a, f(a))$ is a *cusp point* if the tangent line at P is vertical and the function is increasing on one side of the point P and decreasing on the other side.

$$f'(a) = DNE$$



Ex 2. Find the numbers x where the function $y = f(x)$ (see the graph below) is not differentiable and explain why.



The function $y = f(x)$ is not differentiable at:

Reading: Nelson Textbook, Pages 65-72
Homework: Nelson Textbook: Page 73 #1, 6, 7b, 9, 14, 16, 19

