2.1 Derivative Function

**A Derivative Function**
Given a function \( y = f(x) \), the derivative function of \( f \) is a new function called \( f' \) (f prime), defined at \( x \) by:

\[
f'(x) = \lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h}
\]

**B Differentiability**
A function \( y = f(x) \) is called **differentiable** at \( x \) if \( f'(x) \) exists. A function \( y = f(x) \) is differentiable over an open interval \((a, b)\) if the function is differentiable at every number in that interval.

Note: The domain of derivative function \( f' \) is a subset of the domain of the original function \( f \): \( D_f \subset D_{f'} \). So a function is defined over \( D_f \) but is differentiable over \( D_{f'} \).

**C Interpretations of Derivative Function**
1. The **slope of the tangent line** to the graph of \( y = f(x) \) at the point \( P(a, f(a)) \) is given by \( m = f'(a) \).
2. The **instantaneous rate of change** in the variable \( y \) with respect to the variable \( x \), where \( y = f(x) \), at \( x = a \) is given by:
   \[ IRC = f'(a) \]

**D Notations and Reading**
- \( y' = f'(x) \) [Lagrange Notation]
  Read: “y prime” or “f prime at x”
- \( \frac{dy}{dx} = \frac{d}{dx} f(x) \) [Leibnitz Notation]
  Read: “dee y by dee x”
- \( f'(a) = \frac{dy}{dx}{x=a} \)
  Read: “f prime at a, dee y by dee x at x equals a”

**E First Principles**
**Differentiation** is the process to find the derivative function for a given function.

**First Principles** is the process of differentiation by computing the limit:

\[
f'(x) = \lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h}
\]

**Ex 1.** Use first principles to differentiate each function.

a) \( f(x) = 2x - x^3 \)

b) \( f(x) = \frac{-3}{x^2} \)

c) \( f(x) = \sqrt{ax + b} \)
**G Non-Differentiability**
A function is **not differentiable** at $x = a$ if $f'(a)$ does not exist.

**Notes:**
1. If a function $f$ is **not continuous** at $x = a$ then the function $f$ is **not differentiable** at $x = a$.
2. If a function $f$ is **continuous** at $x = a$ then the function $f$ **may be** or **not** differentiable at $x = a$.

**H Differentiability Point**
If the function $y = f(x)$ is **differentiable** at $x = a$ then the tangent line at $P(a, f(a))$ is **unique** and **not vertical** (the slope of the tangent line is not $\infty$ or $-\infty$).

**I Corner Point**
$P(a, f(a))$ is a **corner point** if there are **two** distinct tangent lines at $P$, one for the left-hand branch and one for the right-hand branch. For example:

\[
 f(x) = \begin{cases} 
 f_1(x), & x < a \\
 f_2(x), & x > a 
\end{cases} \quad \text{and} \quad f'_1(a) \neq f'_2(a)
\]

**J Infinite Slope Point**
$P(a, f(a))$ is an **infinite slope point** if the tangent line at $P$ is vertical and the function is increasing or decreasing in the neighborhood at the of the point $P$.

\[
 f'(a) = \infty \text{ or } f'(a) = -\infty
\]

**K Cusp Point**
$P(a, f(a))$ is a **cusp point** if the tangent line at $P$ is vertical and the function is increasing on one side of the point $P$ and decreasing on the other side.

\[
 f'(a) = \text{DNE}
\]

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**Reading:** Nelson Textbook, Pages 65-72

**Homework:** Nelson Textbook: Page 73 #1, 6, 7b, 9, 14, 16, 19