### 1.6 Continuity

#### A Continuity
A function \( y = f(x) \) is **continuous** at a number \( a \) if
\[
\lim_{x \to a} f(x) = f(a)
\]
Note: A function is continuous at \( a \) if the following three conditions are met:
1. \( f(a) \) exists
2. \( \lim_{x \to a} f(x) \) exists
3. \( f(a) \) and \( \lim_{x \to a} f(x) \) are equal.

Note: A function is continuous if the graph can be drawn without lifting the pen from paper.

#### B Discontinuity
If \( y = f(x) \) is not continuous at \( a \) then we say:
- \( y = f(x) \) is **discontinuous** at \( a \)
- \( y = f(x) \) has a discontinuity at \( a \)

Note: There are three types of discontinuity:
- **removable or point discontinuity**
- **jump discontinuity**
- **infinite discontinuity** (break)

#### C Removable or Point Discontinuity
A function \( y = f(x) \) has a **removable or point discontinuity** at \( a \) if:
1. \( \lim_{x \to a} f(x) \) exists
2. \( f(a) \) Does Not Exists

or \( \lim_{x \to a} f(x) \neq f(a) \)

Note: A removable or point discontinuity can be removed by redefining the function at \( a \) as
\[
\mathbf{def} \quad f(a) = \lim_{x \to a} f(x)
\]

Ex 1. Redefine \( y = f(x) = \frac{x^2 - 4}{x - 2} \) such that \( y = f(x) \) is to be continuous everywhere (at any number). Graph the old and the new function.
### D Jump Discontinuity
A function $y = f(x)$ has a jump discontinuity at $a$ if the left-side and the right-side limits exist but they are not equal:

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

Ex 2. Analyze the continuity of the function $y = f(x) = \frac{|x - 3|}{x - 3}$ at $x = 3$. Graph the function to illustrate the situation.

![Jump Discontinuity Graph](image)

### E Infinite Discontinuity
A function $y = f(x)$ has an infinite discontinuity at $a$ if at least one of the left-side or the right-side limits is unbounded (approaches to $\infty$ or $-\infty$).

$$\lim_{x \to a^-} f(x) = -\infty$$

Ex 3. Analyze the continuity of the function $f(x) = \frac{1}{x}$ at $x = 0$.

$$\lim_{x \to a^+} f(x) = \infty$$

To write $\lim_{x \to a} f(x) = \infty$ is better (more information is included) than to write $\lim_{x \to a} f(x)$ DNE.

![Infinite Discontinuity Graph](image)
Ex 4. The function \( y = f(x) \) is represented graphically in the figure on the right side.

Analyze the continuity of this function at:

<table>
<thead>
<tr>
<th>a) ( x = -3 )</th>
<th>b) ( x = -2 )</th>
<th>c) ( x = -1 )</th>
<th>d) ( x = 0 )</th>
<th>e) ( x = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image.png" alt="Graph" /></td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

F Elementary Functions
Elementary functions (polynomial, power, rational, trigonometric, exponential, and logarithmic) are continuous over their domain.

Ex 5. Analyze the continuity of the function:

\[
 f(x) = \begin{cases} 
 x, & x < 0 \\
 x^2, & 0 \leq x \leq 1 \\
 x^3 + 1, & x > 1 
\end{cases}
\]

Ex 6. For what value of the constant \( c \) is the function

\[
 f(x) = \begin{cases} 
 x + c, & x < 2 \\
 cx^2 + 1, & x \geq 2 
\end{cases}
\]

continuous at any number (everywhere)?

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**Reading:** Nelson Textbook, Pages 48-51

**Homework:** Nelson Textbook: Page 51 #4a, 5c, 7, 12, 15, 16, 17