1.5 Tangent and Normal Lines

A. Lines

(*Slope y-intercept Equation*) The equation of a *non vertical line* in *slope y-intercept* form is:

$$y = mx + b$$

where m is the slope of the line and b is the y-intercept.

(*Slope Formula*) If two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are given on the line, then the slope is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

(*Slope-Point Equation*) If given the slope of the line and a point $P_1(x_1, y_1)$ on the line, then the equation of the line is given by:

$$y - y_1 = m(x - x_1)$$

B. Parallel and Perpendicular Lines

(*Parallel Lines*) The slopes of two parallel lines are equal:

$$m_1 = m_2$$

(*Perpendicular Lines*) The slopes of two perpendicular lines are negative reciprocal:

$$m_1 = -\frac{1}{m_2} \qquad \text{or} \qquad m_1 m_2 = -1$$

C. Tangent Line

Consider a function $y = f(x)$ and a point $P(a, f(a))$ on its graph.

(*Slope of Tangent Line)* The slope of the tangent line passing through the point $P(a, f(a))$ (called the point of tangency) is given by:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

or:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(*Equation of Tangent Line*) The equation of tangent line passing through the point $P(a, f(a))$ and having the slope m is:

$$y - f(a) = m(x - a)$$

D. Normal Line

Consider a function $y = f(x)$ and a point $P(a, f(a))$ on its graph.

(*Normal Line*) The normal line passing through the point $P(a, f(a))$ is perpendicular on the tangent line passing through the same point.

(*Equation of the Normal Line*) The equation of normal line passing through the point $P(a, f(a))$ where the slope of the tangent line is m is given by:

$$y - f(a) = -\frac{1}{m}(x - a)$$

Practice Questions

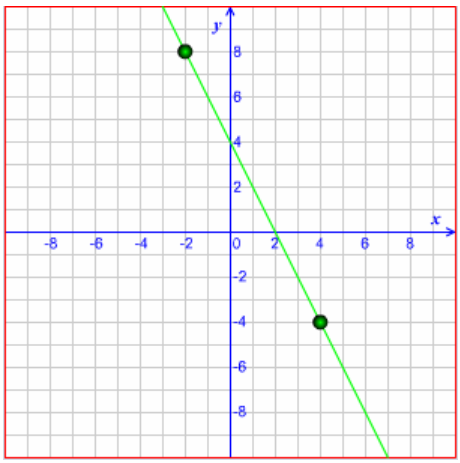
A. Lines

1. Find the slope and the y-intercept for each of the following lines.

a) $y = 3$ b) $y = -x + 2$

c) $y = 3x - 2$ d) $2x - 3y + 1 = 0$

2. Find the slope and the y-intercept for the following line:



3. Find the equation of a line with a slope $m = -3$ that passes through the point $P(-2,5)$.

B. Parallel and Perpendicular Lines

1. Find the equation of a line passing through the point $P(1,2)$ and perpendicular on the line $y = -2x + 3$.

C. Tangent Line

1. For each function, calculate first the slope of the tangent line with the formula:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

then find the equation of the tangent line at the given point.

a) $f(x) = x^2 + x$, at $P(-1,0)$

b) $f(x) = \frac{x-1}{x+1}$, at $P(1,0)$

c) $f(x) = \frac{2}{x^2+1}$, at $P(-1,1)$

d) $f(x) = \sqrt{x-1}$, at $P(2,1)$

e) $f(x) = x + \sqrt{x}$, at $P(1,2)$

f) $f(x) = \sqrt[3]{x}$, at $P(1,1)$

g) $f(x)=\dfrac{1}{\sqrt{x+1}}$, at $P(0,1)$

2. For each case find the slope of the tangent line at the general point $P(a, f(a))$.

a) $f(x)=x^2$ b) $f(x)=x^3$

c) $f(x)=x^2-2x+1$ d) $f(x)=\dfrac{1}{x}$

e) $f(x)=\dfrac{1}{x^2+1}$ f) $f(x)=\sqrt{x}$

g) $f(x)=\sqrt{x^2+x}$ h) $f(x)=\dfrac{1}{\sqrt{x+1}}$

i) $f(x)=\dfrac{x+1}{x+2}$

3. Find the equation of the tangent line to $y=x^3$ that is parallel to the line $y=3x-2$.

4. Find the point(s) on the graph of $y=x^3$ at which the tangent line is perpendicular to the line $3x+36y=4$.

5. Find the equation of the tangent line to $y=x^2$ passing through the point $P(0,-4)$.

D. Normal Line

1. For each function, calculate first the slope of the tangent line with the formula:

$$m=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

then find the equation of the normal line at the given point.

a) $f(x)=x^3-2x$, at $P(0,-2)$

b) $f(x)=\dfrac{x}{x-1}$, at $P(2,1)$

c) $f(x)=\dfrac{2}{x^2-1}$, at $P(2,2/3)$

d) $f(x)=1-\sqrt{x+1}$, at $P(3,-1)$

e) $f(x)=x\sqrt{x}$, at $P(1,1)$

f) $f(x)=\dfrac{1}{\sqrt{x}-1}$, at $P(4,1)$

C.Q. Challenge Questions

1. For each function, find the equation of tangent line at the given point.

a) $f(x)=\sin x$, at $P(0,0)$

b) $f(x)=\cos x$, at $P(\pi/2,0)$

c) $f(x)=\tan x$, at $P(\pi/3,\sqrt{3})$

d) $f(x)=e^x$, at $P(1,e)$

e) $f(x)=2^x$, at $P(2,4)$

f) $f(x)=\ln x$, at $P(1,0)$

g) $f(x)=\log x$, at $P(2,\log 2)$

Answers

A1. a) $m=0, y-\text{int}=3$ b) $m=-1, y-\text{int}=2$
c) $m=3, y-\text{int}=-2$ d) $m=2/3, y-\text{int}=1/3$

A2. $m=-2, y-\text{int}=4$

A3. $y=-3x-1$

B1. $y=(x+3)/2$

C1. a) $m=-1$, $y=-x-1$ b) $m=1/2$, $y=(x-1)/2$
c) $m=1$, $y=x+2$ d) $m=1/2$, $y=x/2$
e) $m=3/2$, $y=(3x+1)/2$ f) $m=1/3$, $y=(x+2)/3$
g) $m=-1/2$, $y=(-1/2)x+1$

C2. a) $2a$ b) $3a^2$ c) $2a-2$ d) $-1/a^2$
e) $-2a/(a^2+1)^2$ f) $1/(2\sqrt{a})$ g) $(2a+1)/(2\sqrt{a^2+a})$
h) $(-1/2)/(a+1)^{3/2}$ i) $1/(a+2)^2$

C3. $y=3x+2$ or $y=3x-2$

C4. $(-2,-8)$ or $(2,8)$

C5. $y=4x-4$ or $y=-4x-4$

D1. a) $m=-2$, $y=(1/2)x-2$ b) $m=-1$, $y=x-1$
c) $m=-8/9$, $y=(9/8)x-19/12$ d) $m=-1/4$,
$y=4x-13$ e) $m=3/2$, $y=(-2/3)x+5/3$
f) $m=-1/4$, $y=4x-15$

CQ1. a) $y=x$ b) $y=-x+\pi/2$ c) $y=4x-4\pi/3+\sqrt{3}$
d) $y=ex$ e) $y=(4\ln 2)x-8\ln 2+4$ f) $y=x-1$
g) $y=(\log 2)(x/2)$

