1.6 Rates of Change

A. Rate of Change

Consider a function $y = f(x)$ and two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on its graph. The *change in the variable* x is defined by:

$$\Delta x = x_2 - x_1$$

The *change in the variable* y is defined by:

$\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$

The *average rate of change* of the function f over the interval $[x_1, x_2]$ is defined by the quotient:

$$ARC = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Geometrically, the average rate of change is the *slope of the secant line* passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

B. Instantaneous Rate of Change

Consider a function $y = f(x)$ and two points $A(a, f(a))$ and $B(x, f(x))$ on its graph.

The *average rate of change* of the function f over the interval $[a, x]$ is defined by the quotient:

$$ARC = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$

As the point B approaches A:

a) the secant line passing through the points A and B approaches to the tangent line passing through the point A

b) the average rate of change approaches the *instantaneous rate of change* at a:

$$IRC = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

c) geometrically, the instantaneous rate of change represents the *slope of the tangent line* at a:

$$IRC = m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

If $x = a + h$ then the instantaneous rate of change or the slope formula changes to:

$$IRC = m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

C. Experimental Data Approach

Experimental data are given as set of ordered pairs (x_1, y_1), (x_2, y_2), (x_3, y_3), …, (x_n, y_n) (in the increasing order of the variable x).

(*Average Rate of Change*) The average rate of change over the interval $[x_i, x_j]$ is given by:

$$ARC = \frac{\Delta y}{\Delta x} = \frac{y_j - y_i}{x_j - x_i}$$

(*Finite Differences*) The following table allows the calculation of the rate of change for all consecutive ordered pairs (process called *numerical derivative*):

Δx	x	y	Δy	$\frac{\Delta y}{\Delta x}$
	x_1	y_1		
	x_2	y_2		
	x_3	y_3		
…	…	…	…	…

(*Instantaneous Rate of Change*) In order to estimate the instantaneous rate of change at x_k, use one of the following methods:

a) plot the ordered pairs, use a smooth curve to join all the points, draw the tangent line at (x_k, y_k) and find its slope (using the geometrical approach)

b) use Excel (or an equivalent software) to get the equation of the trend line and find the IRC using a numerical or algebraic approach

c) estimate the IRC as the ARC for (x_{k-1}, y_{k-1}) and (x_{k+1}, y_{k+1})

d) estimate the IRC as the average of

- the ARC of (x_{k-1}, y_{k-1}) and (x_k, y_k)
- the ARC of (x_k, y_k) and (x_{k+1}, y_{k+1})

D. Graphical Approach

In order to estimate the instantaneous rate of change at a point $P(x, y)$:

a) draw the tangent line at P

b) find the coordinates of two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the tangent line

c) estimate the IRC using the formula:

$$IRC \cong \frac{y_2 - y_1}{x_2 - x_1}$$

E. Numerical Approach

In order to get a numerical estimation of the IRC at a, use the formula:

$$IRC \cong \frac{f(a + h) - f(a)}{h}$$

by taking $h = 0.1, \quad 0.01, \quad 0.001, \quad 0.0001, \ldots$ and observing the trend.

Ex:

F. Algebraic Approach

If the function is defined by an expression, then the ARC is given by:

$$ARC = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$

And the IRC is given by one of the following relations: $IRC = m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ (1)

or

$$IRC = m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad (2)$$

G. Velocity

Consider a point object moving along a straight line. The *position function* $s(t)$ determines the position of the object at the moment of time t.

The *Average Velocity* over the interval $[t_1, t_2]$ is defined by:

$$v_{average} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

The *Instantaneous Velocity* at the moment of time $t = a$ is defined by one of the following relations:

$$v = \lim_{t \to a} \frac{s(t) - s(a)}{t - a} \qquad (1)$$

or

$$v = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} \qquad (2)$$

Geometrically, the instantaneous velocity is equal to the slope of the tangent line to the graph $s \text{ versus } t$ at the number $t = a$.

Practice Questions

C. Experimental Data Approach

1. An object is launched vertically up and its height is recorded after each second. The experimental data are provided in the next table.

Δt	t (seconds)	h (meters)	Δh	$\Delta h / \Delta t$	$\Delta^2 h$
	0	0			
	1	95.1			
	2	180.4			
	3	255.9			
	4	321.6			
	5	377.5			
	6	423.6			
	7	459.9			
	8	486.4			
	9	503.1			
	10	510			
	11	507.1			
	12	494.4			
	13	471.9			
	14	439.6			
	15	397.5			
	16	345.6			
	17	283.9			
	18	212.4			
	19	131.1			
	20	40			

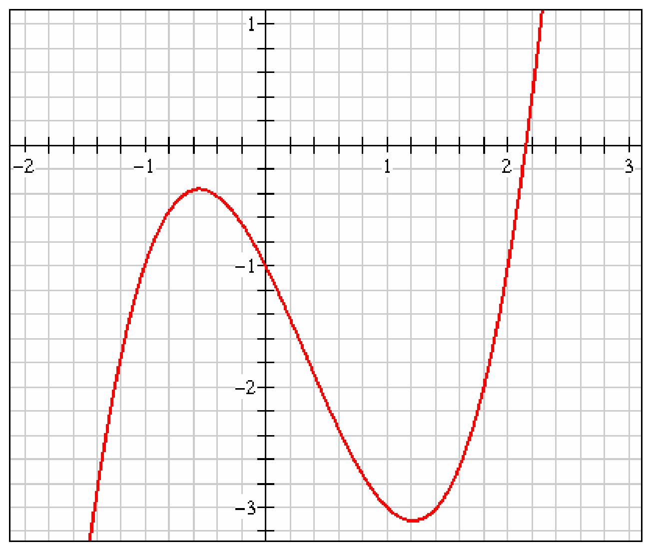
a) estimate the initial height
b) estimate the maximum height
c) find the ARC over the interval $[0,1]$
d) find the ARC over the interval $[1,4]$
e) estimate the IRC at $t = 5$
f) estimate the IRC at $t = 15$

2. Complete the previous table data and answer to the following questions.
a) What is the meaning of the quantity $\Delta h / \Delta t$?
b) When is the quantity $\Delta h / \Delta t$ positive?
c) When is the quantity $\Delta h / \Delta t$ negative?
d) What happen when $\Delta h / \Delta t$ changes its sign?
e) What kind of relation is $h(t)$?

D. Graphical Approach

1. A function is defined by the following graph.

a) find the ARC over the interval $[-1,1]$
b) find the ARC over the interval $[0,2]$
c) find the IRC at $x = -1$
d) find the IRC at $x = 0$
e) find the IRC at $x = 2$



E. Numerical Approach

1. For each function, find the ARC over the given interval.

a) $f(x) = x^2 - 2x$, $[1,3]$

b) $f(x) = \dfrac{\sqrt{x} - 1}{x + 2}$, $[0,1]$

c) $f(x) = \sin x - \cos x$, $[0, \pi / 2]$

d) $f(x) = e^x + \ln x$, $[1,2]$

2. For each function, estimate the IRC at the given number by taking $h = 0.1, \quad 0.01, \quad 0.001, \ldots$ and observing the trend.

a) $f(x) = x^3$, at $a = 1$

b) $f(x)=\frac{2}{x^2}$, at $a=2$

c) $f(x)=x\sin x$, at $a=\pi/2$

d) $f(x)=xe^x$, at $a=0$

e) $f(x)=x\ln x$, at $a=1$

G. Velocity

1. For each case, find the average velocity over the given interval.

a) $s(t)=t^2+t$, $[0,2]$

b) $s(t)=t^3-t^2$, $[1,2]$

c) $s(t)=t+\frac{1}{t}$, $[1,4]$

d) $s(t)=\sqrt{t+1}$, $[0,3]$

2. For each case, find the instantaneous velocity at the given moment of time.

a) $s(t)=2t^2-t$, at $t=1$

b) $s(t)=2t^3-3t$, at $t=0$

c) $s(t)=\frac{t}{t+1}$, at $t=2$

d) $s(t)=\sqrt{t^2+3}$, at $t=1$

Answers

C1. a) 0 b) 510 c) $95.1m/s$ d) $75.5m/s$ e) $51m/s$ f) $-47m/s$
C2. a) speed b) over $[0,10]$ c) over $[10,20]$ d) h is maximum e) quadratic
D1. a) -1 b) 0 c) -1 d) -2 e) 6
E1. a) 2 b) $1/2$ c) $4/\pi$ d) $e^2+\ln 2-e$
E2. a) 3 b) -0.5 c) 1 d) 1 e) 1

