2.4 Quotient Rule

A. Quotient Rule

(*Product Rule*) If f and g are differentiable at x and $g(x) \neq 0$, then so is $\frac{f}{g}$ and (in Lagrange notation)

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

or (in Leibniz notation):

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

(*Reciprocal Rule*) If f is differentiable at x and $f(x) \neq 0$, then so is $\frac{1}{f}$ and

$$\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{[f(x)]^2}, \quad f(x) \neq 0$$

or

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$$

or

$$\frac{d}{dx}\frac{1}{f(x)} = -\frac{\frac{d}{dx}f(x)}{[f(x)]^2}, \quad f(x) \neq 0$$

Practice Questions

A. Quotient Rule

1. Differentiate, then simplify.

a) $f(x) = \frac{x}{x+1}$ b) $f(x) = \frac{x}{x^2+1}$

c) $f(x) = \frac{x^2}{x-1}$ d) $f(x) = \frac{x^2-1}{x^2+1}$

e) $f(x) = \frac{\sqrt{x}}{\sqrt{x}-2}$ f) $f(x) = \frac{x}{1+\sqrt[3]{x}}$

2. Differentiate.

a) $f(x) = \frac{1}{x+2}$ b) $f(x) = \frac{1}{\sqrt{x}-1}$

c) $f(x) = \frac{1}{x^2+1}$ d) $f(x) = \frac{1}{(x+1)^3}$

3. Differentiate.

a) $f(x) = \frac{\sin x}{\cos x}$ b) $f(x) = \frac{\sin x}{e^x}$

c) $f(x) = \frac{2^x}{\cos x}$ d) $f(x) = \frac{\ln x}{e^x}$

e) $f(x) = \frac{3^x}{\log x}$ f) $f(x) = \frac{x}{\cos x}$

g) $f(x) = \frac{\sqrt{x}}{\ln x}$ h) $f(x) = \frac{\ln x}{\sin x}$

4. Find the equation of the tangent line to the each curve that passes through the given point.

a) $y = \frac{1}{1+x^3}$ at $P(1,1/2)$

b) $y = \frac{3x}{x^2+2}$ at $P(1,1)$

c) $y = \frac{\sqrt{x}-1}{\sqrt{x}+1}$ at $P(1,0)$

5. Find the point(s) where the tangent line to the curve $y = \frac{x^2-2x+1}{x^2+x+1}$ is (are) parallel to the x-axis.

6. For each case, find the intervals where the function is increasing or decreasing.

a) $f(x) = \frac{1}{x+1}$ b) $f(x) = \frac{x-2}{x+2}$

c) $f(x) = \frac{|x|}{|x|+1}$ d) $f(x) = \frac{x^2}{2x+1}$

e) $f(x) = \frac{x^2+1}{x+1}$

Challenge Questions

1. Differentiate.

a) $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ b) $f(x) = \frac{\sin x - \cos x}{\sin x + \cos x}$

2. Differentiate.

a) $f(x) = \frac{ax+b}{cx+d}$ b) $f(x) = \frac{1}{ax^2+bx+c}$

c) $f(x) = \left(\frac{x-1}{x+1}\right)^3$

3. Given that $f(2) = 1$, $f'(2) = -1$, $g(2) = 2$, and $g'(2) = -2$ find the values of:

a) $(f/g)'(2)$ b) $(g/f)'(2)$

c) $(1/f)'(2)$ d) $(1/g)'(2)$

e) $[(f-g)/(f+g)]'(2)$ f) $[(1+2f)/(1-3g)]'(2)$

4. Let $f(x)=\frac{x^3}{1+x^2}$. Find the point(s) on the curve $y=f'(x)$ where the tangent line is horizontal.

5. Differentiate.

a) $\frac{x\cos x}{\sin x}$ b) $\frac{\sqrt{x}e^x}{\ln x}$ c) $\frac{\sin x\ln x}{e^x}$

d) $\frac{e^x\sin x}{\ln x\cos x}$ e) $\frac{x^3\sin^2 x}{\cos^2 x}$ f) $\frac{x\sqrt{x}\ln x}{e^x\sin^2 x}$

6. Find the point(s) on the curve $y=\frac{\cos x}{1+\sin^2 x}$ where the tangent line is horizontal.

7. Find the equation of the tangent line to the curve $y=\frac{\ln(x+1)}{e^x+1}$ at the point $(0,0)$.

Answers

A1. a) $f(x)=\frac{1}{(x+1)^2}$ b) $f(x)=\frac{1-x^2}{(x^2+1)^2}$

c) $f(x)=\frac{x^2-2x}{(x-1)^2}$ d) $f(x)=\frac{4x}{(x^2+1)^2}$

e) $f(x)=\frac{-1}{\sqrt{x}(\sqrt{x}-2)^2}$ f) $f(x)=\frac{1+(2/3)\sqrt[3]{x}}{(1+\sqrt[3]{x})^2}$

2. a) $f(x)=\frac{-1}{(x+2)^2}$ b) $f(x)=\frac{-1}{2\sqrt{x}(\sqrt{x}-1)^2}$

c) $f(x)=\frac{-2x}{(x^2+1)^2}$ d) $f(x)=\frac{-3}{(x+1)^4}$ (??)

3. a) $f(x)=\frac{1}{\cos^2 x}$ b) $f(x)=\frac{\cos x-\sin x}{e^x}$

c) $f(x)=\frac{2^x(\ln 2\cos x+\sin x)}{\cos^2 x}$ d) $f(x)=\frac{1/x-\ln x}{e^x}$

e) $f(x)=\frac{3^x(\ln 3\log x-1/(x\ln 10)}{\log^2 x}$

f) $f(x)=\frac{\cos x+x\sin x}{\cos^2 x}$ g) $f(x)=\frac{(1/2)\ln x-1}{\sqrt{x}\ln^2 x}$

h) $f(x)=\frac{(1/x)\sin x-\ln x\cos x}{\sin^2 x}$

4. a) $y=(-3x+5)/4$ b) $y=(x+2)/3$ c) $y=(x-1)/4$

5. $P(-1,4/3)$ and $Q(1,0)$

6. a) $f'(x)=\frac{-1}{(x+1)^2}<0$, f is decreasing everywhere

b) $f(x)=\frac{4}{(x+2)^2}>0$, f is increasing everywhere

c) $f'(x)=\begin{cases}1/(x+1)^2>0, & x>0\\ -1/(x-1)^2<0, & x<0\end{cases}$, f is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$

d) $f'(x)=\frac{2x(x+1)}{(2x+1)^2}$, f is increasing on $(-\infty,-1)\cup(0,\infty)$ and decreasing on $(-1,0)$

e) $f'(x)=\frac{x^2+2x-1}{(x+1)^2}$, f is decreasing on $(-1-\sqrt{2},-1+\sqrt{2})$ and is increasing on $(-\infty,-1-\sqrt{2})\cup(-1+\sqrt{2},\infty)$

CQ1. a) $f'(x)=\frac{4}{(e^x+e^{-x})^2}$ b) $f'(x)=\frac{4\sin x\cos x}{(\sin x+\cos x)^2}$

2. a) $f'(x)=\frac{ad-bc}{(cx+d)^2}$ b) $f'(x)=\frac{-(2ax+b)}{(ax^2+bx+c)^2}$

c) $f'(x)=\frac{6(x-1)^2}{(x+1)^4}$

3. a) 0 b) 0 c) 1 d) $1/2$ e) 0 f) $-8/25$

4. $(0,0)$ or $(\pm\sqrt{3},9/8)$

5. a) $\frac{\sin x\cos x+x(\sin^2 x-\cos^2 x)}{\sin^2 x}$

b) $\frac{e^x[(x+1/2)\ln x-1]}{\sqrt{x}\ln^2 x}$

c) $\frac{\ln x(\cos x-\sin x)+(\sin x)/x}{e^x}$

d) $\frac{e^x[1+\sin x\cos x(\ln x-1/x)]}{\ln^2 x\cos^2 x}$

e) $\frac{x^2\sin x(3\sin x\cos x+2x)}{\cos^3 x}$

f) $\frac{\sqrt{x}[(3/2)\ln x+1)\sin x-x\ln x(\sin x+2\cos x)]}{e^x\sin^3 x}$

6. $y'=\frac{\sin x(\sin^2 x-3)}{(1+\sin^2 x)^2}$. The point are $(2n\pi,1)$ and $((2n+1)\pi,-1)$ where n is integer.

7. $y=x/2$

