

2.3 Product Rule

A. Product Rule

(*Product Rule*) If f and g are differentiable at x , then so is fg and (in Lagrange notation)

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$(fg)' = f'g + fg'$$

or (in Leibniz notation):

$$\frac{d}{dx}(f(x)g(x)) = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$

(*Generalization of the Product Rule*) If f, g and h are differentiable at x , then so is fgh and:

$$(fgh)' = f'gh + fg'h + fgh'$$

B. Generalized Power Rule

(*Generalized Power Rule*) If $f(x)$ is differentiable at x , then so is $(f(x))^n$ (n is a positive integer) and:

$$[(f(x))^n]' = n(f(x))^{n-1}f'(x)$$

or:

$$\frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)$$

Practice Questions

A. Product Rule

1. Differentiate using two methods and compare results.

Method 1. Multiply first and then differentiate
Method 2. Use the product rule to differentiate

a) $f(x) = (x-1)(x+2)$ b) $f(x) = \sqrt{x}(\sqrt{x}-1)$

c) $f(x) = (2x^2 - x)(x^2 + 2x)$

d) $f(x) = \left(x + \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)$

e) $f(x) = (\sqrt{x} + \sqrt[3]{x})(\sqrt{x} - \sqrt[3]{x})$

2. Differentiate. Do not simplify.

a) $f(x) = (x+1)(x-2)(x+3)$

b) $f(x) = (x^2 + 1)(x^2 - x)(x^3 + 2x)$

c) $f(x) = \sqrt{x}(x + \sqrt{x})(1 - \sqrt[3]{x})$

3. Determine the equation of the tangent line to each curve at the given value of x .

a) $f(x) = (x^2 + 1)(x^3 - 2)$ at $x = 0$

b) $f(x) = (\sqrt{x} - 1)(x + \sqrt{x})$ at $x = 1$

c) $f(x) = \left(x + \frac{1}{x}\right)\left(x^3 + \frac{1}{x^2}\right)$ at $x = -1$

4. Differentiate.

a) $x \sin x$ b) $\sqrt{x} \cos x$ c) $\sin x \cos x$

d) xe^x e) $x^2 2^x$ f) $\sqrt{x} \ln x$

g) $x \log x$ h) $x^2 \log_2 x$ i) $e^x \ln x$

j) $2^x \sin x$ k) $\ln x \sin x$ l) $e^x 2^x$

5. Differentiate.

a) $\frac{\sin x}{x}$ b) $\frac{\cos x}{\sqrt{x}}$ c) $\frac{e^x}{x}$

d) $\frac{\ln x}{x}$ e) $\frac{2^x}{x^2}$ f) $\frac{\log_2 x}{\sqrt[3]{x}}$

6. Differentiate.

a) $x \sin x \ln x$ b) $\sqrt{x}e^x \cos x$

c) $2^x \ln x \sin x$ d) $\frac{(\sin x)e^x}{x}$

B. Generalized Power Rule

1. Differentiate using the power-chain rule.

a) $f(x) = (2x-1)^2$ b) $f(x) = (x^2 - x)^2$

c) $f(x) = (x - \sqrt{x})^2$ d) $f(x) = (x+2)^3$

e) $f(x) = \left(x + \frac{1}{x}\right)^4$ f) $f(x) = (x^2 + x + 1)^{100}$

2. Differentiate.

a) $f(x) = \sin^3 x$ b) $f(x) = \cos^4 x$

c) $f(x) = (e^x)^5$ d) $f(x) = (3^x)^5$

e) $f(x) = (\ln x)^5$ f) $f(x) = (\log x)^2$

g) $f(x) = (\log_3 x)^4$

Challenge Questions

1. Evaluate $\lim_{x \rightarrow 1} \frac{x^{1000} - 1}{x - 1}$.

2. Given that $f(1) = 2$, $f'(1) = -3$, $g(1) = -1$, and $g'(1) = -2$ find the values of:

a) $(fg)'(1)$ b) $\frac{d}{dx}[f(x)]^2 \Big|_{x=1}$

c) $\frac{d}{dx}f(x)[g(x)]^2 \Big|_{x=1}$ d) $\frac{d}{dx}[f(x)]^2[g(x)]^3 \Big|_{x=1}$

3. Determine the equation of the tangent line to each curve at the given value of x .

- a) $f(x) = (x + \sqrt{x})(x^2 - 2)(x + 1)$ at $x = 1$
 b) $f(x) = 2x^2(x^2 + 2x)(x - 1)$ at $x = -1$
 c) $f(x) = 2\sqrt{x}\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x}\right)$ at $x = 1$
 d) $f(x) = 3x\left(x^{\frac{2}{3}} + 3x^{\frac{1}{3}}\right)(-4x^2 + 5)$ at $x = -1$

4. Let $f(x) = \sqrt[3]{x}(x^2 - 3x + 2)$. Find:

- a) $f'(x)$
 b) the x -values where the tangent is horizontal
 c) the x -values where the tangent is vertical

5. Differentiate.

- a) $\sin^2 x \cos^3 x$ b) $x^3 \ln^2 x$
 c) $\sin^3 x \log^2 x$ d) $(e^x)^3 \sin^2 x$
 e) $(\sqrt{x} \sin x)^2 \left(\frac{\log x}{x}\right)^3$

Answers

- A1. a)** $f'(x) = 2x + 1$ **b)** $f'(x) = 1 - 1/(2\sqrt{x})$
 c) $f'(x) = 8x^3 + 9x^2 - 4x$
 d) $f'(x) = 1 + 3x^2 + 1/x^2 + 3/x^4$ **e)** $f'(x) = 1 - 2/(3\sqrt[3]{x})$
2. a) $f(x) = (x - 2)(x + 3) + (x + 1)(x + 3) + (x + 1)(x - 2)$
 b)
 $f(x) = (2x)(x^2 - x)(x^3 + 2x) +$
 $+(x^2 + 1)(2x - 1)(x^3 + 2x) + (x^2 + 1)(x^2 - x)(3x^2 + 2)$
 c)
 $f(x) = \left(\frac{1}{2\sqrt{x}}\right)(x + \sqrt{x})(1 - \sqrt[3]{x}) + \sqrt{x}\left(1 + \frac{1}{2\sqrt{x}}\right)(1 - \sqrt[3]{x}) +$
 $+\sqrt{x}(x + \sqrt{x})\left(\frac{-1}{3\sqrt[3]{x^2}}\right)$
3. a) $y = -2$ **b)** $y = x - 1$ **c)** $y = -10x - 10$
4. a) $\sin x + x \cos x$ **b)** $\left(\frac{1}{2\sqrt{x}}\right)\cos x - \sqrt{x} \sin x$
 c) $\cos^2 x - \sin^2 x$ **d)** $e^x + xe^x$ **e)** $(2x)2^x + x^2 2^x \ln 2$
 f) $\left(\frac{1}{2\sqrt{x}}\right)\ln x + \frac{\sqrt{x}}{x}$ **g)** $\log x + 1/\ln 10$
 h) $2x \log_2 x + x/\ln 2$ **i)** $e^x \ln x + e^x/x$
 j) $(\ln 2)2^x \sin x + 2^x \cos x$ **k)** $\frac{\sin x}{x} + \ln x \cos x$
 l) $(1 + \ln 2)(2e)^x$
5. a) $\frac{\cos x}{x} - \frac{\sin x}{x^2}$ **b)** $\frac{-\sin x}{\sqrt{x}} - \frac{\cos x}{2x\sqrt{x}}$
 c) $\frac{e^x}{x} - \frac{e^x}{x^2}$ **d)** $\frac{1}{x^2} - \frac{\ln x}{x^2}$ **e)** $\frac{(\ln 2)2^x}{x^2} - \frac{2^{x+1}}{x^3}$
 f) $\frac{1}{(\ln 2)x\sqrt[3]{x}} - \frac{\log_2 x}{3x\sqrt[3]{x}}$

- 6. a)** $\sin x \ln x + x \cos x \ln x + \sin x$
 b) $\frac{1}{2\sqrt{x}}e^x \cos x + \sqrt{x}e^x \cos x - \sqrt{x}e^x \sin x$

- c) $2^x \ln 2 \ln x \sin x + \frac{2^x \sin x}{x} + 2^x \ln x \cos x$
 d) $-\frac{(\sin x)e^x}{x^2} + \frac{(\cos x)e^x}{x} + \frac{(\sin x)e^x}{x}$

B1. a) $f'(x) = 4(2x - 1)$ **b)** $f'(x) = 2(x^2 - x)(2x - 1)$

c) $f'(x) = (x - \sqrt{x})(2 - 1/\sqrt{x})$ **d)** $f'(x) = 3(x + 2)^2$

e) $f'(x) = 4\left(x + \frac{1}{x}\right)^3 \left(1 - \frac{1}{x^2}\right)$

f) $f(x) = 100(x^2 + x + 1)^{99}(2x + 1)$

2. a) $f'(x) = 3 \sin^2 x \cos x$ **b)** $f'(x) = -4 \cos^3 x \sin x$

c) $f'(x) = 5(e^x)^5$ d) $f'(x) = 5(3^x)^5 \ln 3$

e) $f'(x) = 5(\ln x)^4/x$ f) $f'(x) = 2(\log x)/(x \ln 10)$

g) $f'(x) = 4(\log_3 x)^3/(x \ln 3)$

CQ1. 1000

2. a) -1 **b)** -12 **c)** 5 **d)** -12

3. a) $y = 3x - 7$ **b)** $y = -10x - 6$ **c)** $y = 8x$

d) $y = 41x + 47$

4. a) $f'(x) = \frac{7x^2 - 12x + 2}{3\sqrt[3]{x^2}}$ **b)** $x = \frac{6 \pm \sqrt{22}}{7}$ **c)** $x = 0$

5. a) $2 \sin x \cos^4 x - 3 \sin^3 x \cos^2 x$

b) $3x^2 \ln^2 x + x^2(2 \ln x)$

c) $3 \sin^2 x \cos x \log^2 x + (2 \sin^3 x \log x)/x$

d) $3(e^x)^3 \sin^2 x + 2(e^x)^3 \sin x \cos x$

e) $\frac{-2 \sin^2 x \log^3 x}{x^3} + \frac{2 \sin x \cos x \log^3 x}{x^2} + \frac{3 \sin^2 x \log^2 x}{(\ln 3)x^3}$