3. Optimization

Algorithm for Solving Optimization Problems

1. Understand the problem

2. Draw a diagram

3. Assign variables to quantities involved

4. Write relations between these variables

5. Identify the variable that is minimized or maximized (the dependant variable)

6. Use relations between variables to end with one dependent variable and one independent variable

7. Find extrema for the dependant variable

9. Check if extrema satisfy the conditions of the application

10. Write the conclusion statement

Practice Questions

A. Minimizing/Maximizing the sum or the product

1. Find two positive numbers whose product is 100 and whose sum is a minimum. What is the minimum value of the sum?

2. Find two positive numbers whose sum is 40 and whose product is a maximum. What is the maximum value of the product?

3. Find two numbers whose difference is 150 and whose product is a minimum. What is the minimum value of the product?

4. Find two positive numbers with product 200 such that the sum of one number and twice the second number is as small as possible. What is the minimum value of the sum?

5. Find a positive number such that the sum of the square of the number and its reciprocal is a minimum.

*6. Find the maximum value of $x-2y$ on the unit circle $x^2+y^2=1$.

7. The sum of two non-negative numbers is 16. Find the maximum possible value and the minimum possible value of the sum of their cube roots.

B. Maximizing the area

1. A rectangle has a perimeter of $100m$. What length and width should it have so that its area is a maximum. What is the maximum value of its area?

2. The lifeguard at a public beach has $400m$ of rope available to la out a rectangular restricted swimming area using the straight shoreline as one side of the rectangle.
a) If she wants to maximize the swimming area, what will the dimensions of the rectangle be?
b) To ensure the safety of swimmers, she decides that nobody should be more than $50m$ from the shore. What should the dimensions of the swimming area be with this added restriction?

3. A rectangle is inscribed in a circle of radius $8m$.
a) Find the dimensions of the rectangle that will maximize the area of the rectangle.
b) Find the maximum value of the area.

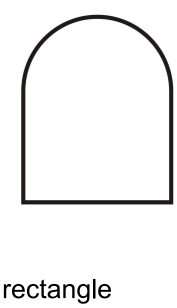
4. A rectangle is inscribed in a semicircle of radius $8m$.
a) Find the dimensions of the rectangle that will maximize the area of the rectangle.
b) Find the maximum value of the area.

5. Find the dimensions of the largest rectangle that can be inscribed in an equilateral triangle with the side length a if one side of the rectangle lies on one side of the triangle.

6. A $1000m$ track is to be build with two straight sides and semicircles at the ends.
a) Find the dimensions of the track that encloses the maximum area.
b) Find the dimensions of the track that encloses the maximum rectangular area.

7. Find the dimensions (length and width) of a rectangle that is inscribed in a right triangle with the base equal to $3m$ and a height equal to $4m$ and has a maximum area.

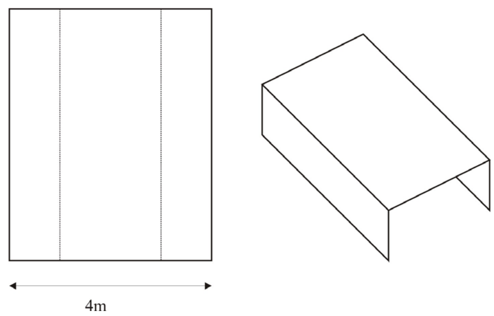
8. A Norman window has a shape represented in the right figure (three sides belongs to a rectangle and the fourth side is a half of a circle). If the total perimeter of the window is $10m$, find the width of the window with the largest area.



9. Find the maximum possible area of a rectangle with diagonals of length $l=16$.

10. A child's play tunnel is to be made from a $4m$ wide sheet of cardboard. The sheet will be folded as

shown. Where should the fold be made in order to maximize the cross-sectional area of the tunnel?



11. Find the area of the largest rectangle that can be inscribed between the x-axis and the graph defined by $y = 9 - x^2$.

12. Find the rectangle of the largest area (with sides parallel to the axes) that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

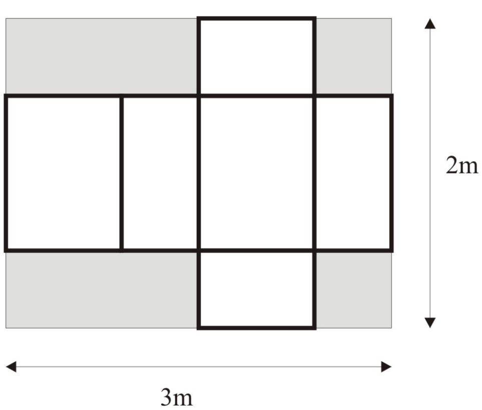
C. Maximizing the volume

1. If $2700cm^2$ of material is available to make a box with a square base and open top, find the dimensions of the box that give the largest volume of the box. What is the maximum value of the volume?

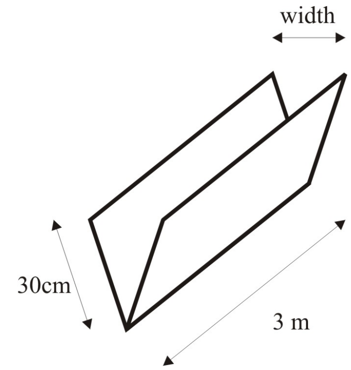
2. A box with an open top is to be constructed from a square piece of cardboard, $2m$ wide, by cutting out a square from each of the four corners and bending up the sides.
a) Find the dimensions of the box corresponding to a maximum volume.
b) Find the maximum value of the volume.

3. Find the dimensions of the largest right-cylinder that can be inscribed in a sphere of radius $10m$.

4. A closed box is to be constructed from a rectangular piece of cardboard $2m$ by $3m$ according to the diagram below. Find dimensions of the box such that the volume of the box is maximum.



5. A $3m$ long feed trough, in the shape of an isosceles triangular prism, is to be made with steels ends and two boards $30cm$ wide. How wide should the top of the trough be to maximize the capacity of the trough?



6. Find the dimensions of the largest right-cylinder that can be inscribed in a cone of radius $R = 3m$ and height $H = 6m$.

7. A cone-shaped paper drink cup is made by removing a sector from a disk of radius R and then joining the two straight edges. Find the maximum capacity of such a cup.

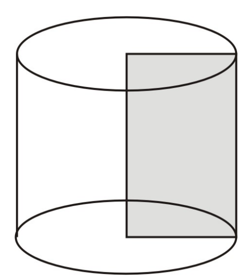
8. A rectangular piece of paper with perimeter $100cm$ is to be rolled to form a cylindrical tube. Find the dimensions of the paper that will produce a tube with maximum volume.

9. What is the maximum possible volume of a cone with slant height $s = 10$?

10. The sum of surface areas of a cube and a sphere is $1000cm^3$. What should their dimensions be to maximize the sum of their surface areas? to minimize it?

11. Of all possible right circular cones that can be inscribed in a sphere of radius R, find the volume of the cone that has maximum volume.

12. A right circular cylinder is generated by rotating a rectangle of perimeter p about one of its sides. What dimensions of the rectangle will generate the cylinder of maximum volume.



D. Maximizing/Minimizing the distance/perimeter

1. Find the points on the parabola $y = 6 - x^2$ that are closest to the point $(0,3)$. What is the minimum value of the distance?

2. Find the point on the line $3y + 2x = 12$ that is closest to the origin.

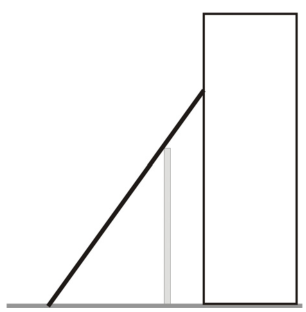
3. Find the shortest distance from the point $(0,2)$ to the hyperbola $x^2 - y^2 = 1$.

4. Find the rectangle of maximum perimeter that can be inscribed in the ellipse $4x^2 + 9y^2 = 36$. The sides of the rectangle are parallel to the axes.

5. Two towers $40ft$ apart are $30ft$ and $20ft$ high respectively. A wire fastened to the top of each tower is guyed to the ground at a point between the towers, and is tightened so that is no sag. How far from the taller tower will the wire touch the ground if the length of the wire is a minimum?

6. A north-south highway intersects an east-west highway at a point P. An automobile crosses P at 10:00 AM, travelling east at a constant speed of $20mi/hr$. At the same instant, another automobile is $2mi$ north of P, travelling south at $50mi/hr$. Find the time at which they are closest to each other, and approximate the minimum distance between the automobiles.

7. A fence $8ft$ tall stands on level ground and runs parallel to a tall building. If the fence is $1ft$ from the building, find the length of the shortest ladder that will extend from the ground over the fence to the wall of the building.



E. Minimizing the cost

1. A cable television is laying cable in an area with underground utilities. Two subdivisions are located on opposite sides of a $100m$ wide river. The company has to connect points P and Q with cable, where P is on the South bank and Q is on the North bank $1200m$ East of P. It cost $\$40/m$ to lay cable underground and $\$80/m$ to lay cable underwater.
a) What is the least expensive way to lay the cable?
b) What is the minimum cost?

2. A farmer wants to fence an area of $240{,}000m^2$ in a rectangular field and divide it in half with a fence parallel to one of the sides of the rectangle. How can be done so as to minimize the cost of the fence?

3. A cylindrical can is to be made to hold $1000cm^3$ (one litre) of oil. Find the radius of the can that will minimize the cost of the metal to make the can.

4. A metal cylinder container with an open top is to hold $1ft^3$. If there is no waste in construction, find the dimensions that require the least amount of material.

5. Corn silos are usually in the shape of a cylinder surmounted by a hemisphere. If the volume of a silo is $1000m^3$, what dimensions of the silo would use the minimum amount of materials?

6. To ensure adequate illumination, the area of the windows needs to be $10m^2$. What dimensions will minimize the amount of outside trim required to frame the window if the window is
a) a rectangle
b) an isosceles triangle
c) a rectangle surmounted by a semicircle
d) a rectangle surmounted by an equilateral triangle

7. A soda cracker package is to be constructed in the shape of a rectangular prism with a square base. The total capacity is $1000cm^3$. Find dimensions that will minimize the cost.

8. A chocolate manufacturer uses an equilateral triangular prism package. If the volume of the prism is $125cm^3$ what dimensions will minimize the cost of the package?

9. A juice can in the shape of a right circular cylinder has a fixed capacity. If the material used for the sides of the can costs $0.5\$/cm^2$ and the material for the top and the bottom costs $0.25\$/cm^2$, find the ratio of the height to the radius that results in a minimum cost.

10. A closed box with a square base is to contain $252cm^3$. The bottom costs $\$5/cm^2$, the top costs $\$2/cm^2$, and the sides cost $\$3/cm^2$. Find the dimensions that will minimize the cost.

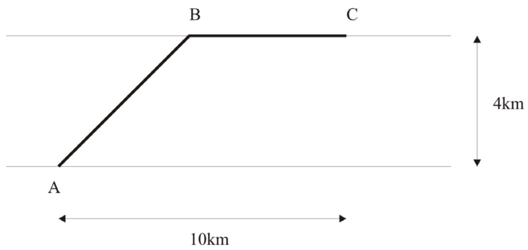
11. A thin-walled cone-shaped cup is to hold $36\pi\ cm^3$ of water when is full. What dimensions will minimize the amount of material needed for the cup?

12. A tank has hemispherical ends and a cylinder center. Find the proportions of the cylinder that will maximize the volume for a given surface area.

F. Minimizing the time

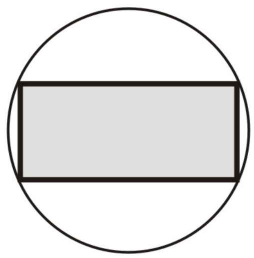
1. A competition has a contestant swimming from a point A on the shore of a lake to a point B on the opposite parallel shore of the lake followed by a running to the finish point C further along the lakeshore. The lake is $4km$ wide and the finish line is $10km$ down the lake. If the contestant can swim at $2km/h$ and run at $10km./h$ determine the position of

the point B that will minimize the total time for the race.



G. Others

1. The strength of a rectangular beam is directly proportional to the product of its width and the square of the depth of a cross section. Find the dimensions of the strongest beam that can be cut from a cylindrical log of radius R.



Answers

A1. 10 and 10, 20
A2. 20 and 20, 400
A3. -75 and 75, -5625
A4. 20 and 10, 40
A5. $1/\sqrt[3]{2}$
A6. $\sqrt{5}$
A7. Maximum: 4, Minimum: $\sqrt[3]{16}$
B1. $10m$ by $10m$, $100m^2$
B2. a) $200m$ by $100m$ b) $300m$ by $50m$
B3. a) $4m$ by $4m$ b) $16m^2$
B4. a) $4m$ by $2m$ b) $8m^2$
B5. $a/2$ by $a\sqrt{3}/4$
B6. a) $l = 0m \text{ and } r = \frac{500}{\pi}m$
b) $l = 250m \text{ and } r = \frac{250}{\pi}m$
B7. $1.5m$ by $2m$
B8. $20/(4+\pi)m$
B9. $l/\sqrt{2}$ by $l/\sqrt{2}$ ($8\sqrt{2}$ by $8\sqrt{2}$)
B10. $1m$ from each edge
B11. $12\sqrt{3}$
B12. $a/\sqrt{2}$ by $b/\sqrt{2}$

C1. $30cm$ by $30cm$ by $15cm$, $13500cm^3$
C2. a) $\frac{4}{3}m$ by $\frac{4}{3}m$ by $\frac{1}{3}m$, b) $\frac{16}{27}m^3$
C3. $r = 5\sqrt{2}, h = 10\sqrt{2}$
C4. $\frac{4+\sqrt{7}}{3}m$ by $\frac{1+\sqrt{7}}{3}m$ by $\frac{5-\sqrt{7}}{6}m$
C5. $30\sqrt{2}cm$
C6. $r = 2R/3 = 2m$, $h = H/3 = 2m$
C7. $\frac{2\pi R^3}{9\sqrt{3}}$
C8. $100/3\ cm$ by $100/6\ cm$
C9. $V = \frac{2\pi s^3}{9\sqrt{3}} = \frac{2000\pi}{9\sqrt{3}}$
C10. To minimize: $r_{spere} = 5\sqrt{10/(\pi+6)}$, $l_{cube} = 2r_{sphere}$
To maximize: $l_{cube} = 0$
C11. $(32/81)\pi R^3$
C12. $r = p/3$, $h = p/6$

D1. $(\sqrt{5/2}, 7/2)$ and $(-\sqrt{5/2}, 7/2)$, $\sqrt{11}/2$
D2. $(3,2)$
D3. $\sqrt{3}$
D4. $18/\sqrt{13}$ by $8/\sqrt{13}$
D5. $24ft$
D6. $1/29 hour$, $\sqrt{16/29}mi$
D7. $5\sqrt{5}ft$

E1. a) underground from P eastward for $1142m$ then underwater to Q (or similar) b) $\$54,928$

E2. $600m$ by $400m$

E3. $\sqrt[3]{\frac{500}{\pi}}cm$
E4. $r = h = 1/\sqrt[3]{\pi}$
E5. $r = h = \sqrt[3]{\frac{600}{\pi}}m \approx 5.76m$
E6. a) $\sqrt{10}m$ by $\sqrt{10}m$
b) $base = \sqrt[4]{\frac{100}{3}}$
c) $width = \sqrt{\frac{80}{4+\pi}}$
d) $width = \sqrt{\frac{40}{6-\sqrt{3}}}$
E7 $10cm$ by $10cm$ by $10cm$
E8. triangle side is $10cm$, height is $\frac{5}{\sqrt{3}}cm$
E9. $1:1$
E10. $6cm$ by $6cm$ by $7cm$
E11. $r = 3\sqrt{3}cm$, $h = 6cm$
E12. a sphere and no cylinder

F1. $9.2km$ from C

G1. $2R/\sqrt{3}$ by $2\sqrt{2}R/\sqrt{3}$

