3.2 Minimum and Maximum Values

A. Minimum and Maximum Values

(*Local Maximum*) A function f has a local (relative) maximum at $x = c$ if $f(x) \le f(c)$ when x is sufficiently close to c (on both sides of c). $f(c)$ is called local (relative) maximum value and $(c, f(c))$ is called local (relative) maximum point.

(*Local Minimum*) A function f has a local (relative) minimum at $x = c$ if $f(x) \ge f(c)$ when x is sufficiently close to c (on both sides of c). $f(c)$ is called local (relative) minimum value and $(c, f(c))$ is called local (relative) minimum point.

(*Global Maximum*) A function f has a global (absolute) maximum at $x = c$ if $f(x) \le f(c)$ for all x in the domain of f. $f(c)$ is called (global or absolute) maximum value and $(c, f(c))$ is called (global or absolute) maximum point.

(*Global Minimum*) A function f has a global (absolute) minimum at $x = c$ if $f(x) \ge f(c)$ for all x in the domain of f. $f(c)$ is called (global or absolute) minimum value and $(c, f(c))$ is called (global or absolute) minimum point.

(*Extremum and Extrema*) An extremum is either a minimum or a maximum (value or point, local or global). Extrema is the plural of extremum.

B. Critical Points

(*Critical Number*) A critical number c is a number in the domain of f where either $f'(c) = 0$ or $f'(c)$ does not exist. The point $(c, f(c))$ is called a critical point.
If $f'(c) = 0$, the critical point is called stationary point.
If $f'(c)$ does not exist, the critical point is called point of nondifferentiability.

(*Fermat's Theorem*) If f has a local extremum (minimum or maximum) at $x = c$, then c is a critical number ($f'(c) = 0$ or $f'(c)$ does not exist).

C. First Derivative Test

(*First Derivative Test*) Let c be a critical point of a continuous function f.
a) If $f'(x)$ changes from positive to negative at c, then f has a local maximum at c.
b) If $f'(x)$ changes from negative to positive at c, then f has a local minimum at c.
c) If $f'(x)$ does not change sign at c, then f has no maximum or minimum at c.

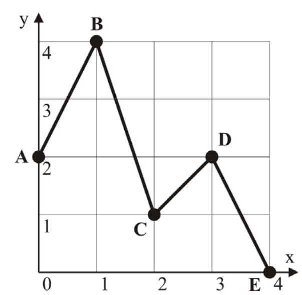
D. Absolute Extrema Algorithm

(*Algorithm*) To find the absolute extrema for a continuous function f on $[a,b]$:
a) identify the critical numbers on $[a,b]$
b) find the values of f at each critical number
c) find the values of f at the endpoints of the interval $f(a)$ and $f(b)$
d) from the values obtained at steps b) and c) the largest value represents the global maximum and the least value represents the global minimum.

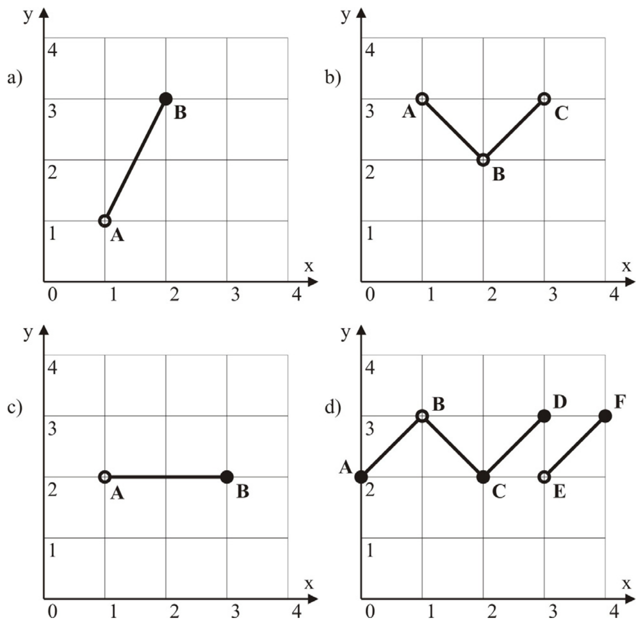
Practice Questions

A. Minimum and Maximum Values

1. A function is defined by its graph. Analyse its extrema.



2. For each case, analyse extrema.



B. Critical Points

1. For each case, find the critical points.

a) $f(x)=x^2+2x$

b) $f(x)=2x^3+3x^2$

c) $f(x)=x^3+6x^2+9x+2$

d) $f(x)=|x|$

e) $f(x)=x^3$

f) $f(x)=\sqrt[3]{x}$

g) $f(x)=\dfrac{1}{x^2}$

h) $f(x)=\dfrac{x}{x+1}$

i) $f(x)=\dfrac{x}{x^2+1}$

j) $f(x)=\begin{cases}0, & x<0\\ 1, & x\ge 0\end{cases}$

k) $f(x)=\dfrac{1}{1+|x|}$

l) $f(x)=x^5+3x^3+1$

m) $f(x)=(x-2)^4$

n) $f(x)=(x^2-4)^2$

o) $f(x)=|x+1|^4$

p) $f(x)=x-\sqrt{x}$

q) $f(x)=(x-1)^{2/3}+2$

r) $f(x)=x\sqrt{x-1}$

s) $f(x)=\sqrt{16-x^2}$

t) $f(x)=\sqrt[3]{x^2-x-2}$

2. For each case, find the critical points.

a) $f(x)=\sin x$

b) $f(x)=\tan x$

c) $f(x)=e^x$

d) $f(x)=\ln x$

e) $f(x)=2^{x^2}$

f) $f(x)=\log(x^2+1)$

g) $f(x)=x\ln x$

h) $f(x)=xe^x$

i) $f(x)=e^{\sqrt{x}}$

j) $f(x)=x^2\ln x^2$

k) $f(x)=x^2e^x$

l) $f(x)=x^2e^{-2x}$

m) $f(x)=\ln x-x$

n) $f(x)=\dfrac{1}{e^x+e^{-x}}$

o) $f(x)=e^{-x^2}$

p) $f(x)=x^2e^{-x^2}$

r) $f(x)=\dfrac{\ln x}{x^2}$

3. Find the value of k if the function $y=x^2+kx+1$ has a local minimum at $x=4$.

4. Find the values of a and b if the function $y=2x^3+ax^2+bx+1$ has a local maximum at $x=-4$ and a local minimum at $x=5$.

C. First Derivative Test

1. For each case, find any local extremum using the first derivative test.

a) $f(x)=1$

b) $f(x)=x$

c) $f(x)=x^2$

d) $f(x)=x^3$

e) $f(x)=\dfrac{1}{x}$

f) $f(x)=\dfrac{1}{x^2}$

g) $f(x)=\sqrt{x}$

h) $f(x)=\dfrac{1}{\sqrt{x}}$

i) $f(x)=x^{1/3}$

j) $f(x)=x^{2/3}$

k) $f(x)=|x|$

2. For each case, find any local extremum using the first derivative test.

a) $f(x)=x^3+3x-1$

b) $f(x)=x^4-2x^2+1$

c) $f(x)=x+\dfrac{1}{x}$

d) $f(x)=x^2(3-2x)$

e) $f(x)=(1-x)^2(1+x)$

f) $f(x)=\dfrac{1+x}{1-x}$

g) $f(x)=\left(\dfrac{1+x}{1-x}\right)^2$

h) $f(x)=\dfrac{2}{x(x-1)}$

i) $f(x)=x-\sqrt{x}$

j) $f(x)=x^2-\sqrt{x}$

k) $f(x)=|x^2-4|$

3. For each case, find k such that f has a relative extremum at $x=3$.

a) $f(x)=x^2+\dfrac{k}{x}$

b) $f(x)=\dfrac{x}{x^2+k}$

D. Absolute Extrema Algorithm

1. For each case, find the absolute extrema (maximum or minimum) points.

a) $f(x)=-2x+3,\quad for\ x\in[-1,2]$

b) $f(x)=x^2-2x,\quad for\ x\in[-2,3]$

c) $f(x)=2x^3+3x^2-12x+1,\quad for\ x\in[-3,2]$

d) $f(x)=3x^4-4x^3,\quad for\ x\in[-1,2]$

e) $f(x)=\sqrt{x-2},\quad for\ x\in[2,6]$

f) $f(x)=|x+2|,\quad for\ x\in[-3,3]$

g) $f(x)=\dfrac{5x}{x+1},\quad for\ x\in[0,4]$

h) $f(x)=\frac{9}{x^2-9}, \quad for\ x\in[-2,1]$

i) $f(x)=\sqrt[3]{x}, \quad for\ x\in[-1,8]$

j) $f(x)=x+\frac{4}{x}, \quad for\ x\in[1,4]$

2. For each case, find the absolute extrema (maximum or minimum) points.

a) $f(x)=\cos x, \quad for\ x\in[-\pi/2,2\pi]$

b) $f(x)=2^x, \quad for\ x\in[-2,2]$

c) $f(x)=x\log x, \quad for\ x\in[0,10]$

d) $f(x)=x-e^x, \quad for\ x\in[-1,2]$

e) $f(x)=xe^{-x}, \quad for\ x\in[-1,2]$

f) $f(x)=\ln(x^2+1), \quad for\ x\in[-3,2]$

g) $f(x)=\frac{e^x}{x^2}, \quad for\ x\in[1,4]$

h) $f(x)=x+\sin x, \quad for\ x\in[0,2\pi]$

Challenge Questions

1. Find the critical points for:

a) $f(x)=x^{2/3}(x-1)^2$

b) $f(x)=(x+5)^2(x-4)^{1/3}$

c) $f(x)=(2x-5)\sqrt{x^2-4}$

d) $f(x)=x^2(2x-5)^{1/3}$

e) $f(x)=3x^{5/3}-15x^{2/3}$

f) $f(x)=\frac{x^4+1}{x^2+1}$

g) $f(x)=x^4-8x^3+22x^2-24x+4$

2. For each case, find any local extremum using the first derivative test.

a) $f(x)=|x^2-4|$

b) $f(x)=x^3(1-x)^2$

c) $f(x)=(1-x)(1+x)^3$

d) $f(x)=\frac{x^2}{1+x}$

e) $f(x)=\frac{|x|}{1+|x|}$

f) $f(x)=|x-1||x+2|$

g) $f(x)=x\sqrt[3]{1-x}$

h) $f(x)=x^2\sqrt{1+x}$

i) $f(x)=\frac{1}{x+1}-\frac{1}{x-2}$

j) $f(x)=x^{7/3}-7x^{1/3}$

k) $f(x)=\frac{x^3}{1+x}$

l) $f(x)=x^{2/3}(x-1)^3$

m) $f(x)=\sqrt[3]{x}-\sqrt[3]{x^2}$

n) $f(x)=\sqrt[3]{x^3-9x}$

o) $f(x)=x\sqrt{8-x^2}$

3. For each case, find the absolute extrema (maximum or minimum) points.

a) $f(x)=|x^2-2x|, \quad for\ x\in[-1,2]$

b) $f(x)=(x-1)^2(x-2)^3, \quad for\ x\in[0,2]$

c) $f(x)=x^2(x-7)^{1/3}, \quad for\ x\in[-1,8]$

d) $f(x)=\frac{x^2-1}{x^2+1}, \quad for\ x\in[-1,3]$

4. Let $f(x)=ax^4+bx^2+cx+d$. Find a,b,c,d such that f has a local maximum at $(0,-6)$ and a local minimum at $(1,-8)$.

5. Find the values of a,b,c and d such that the cubic function $f(x)=ax^3+bx^2+cx+d$ has a local maximum at $(-2,1)$ and a local minimum at $(1,-4)$.

6. Find the values of a,b,c and d such that the quartic function $f(x)=ax^4+bx^2+cx+d$ has a local maximum at $(1,5)$ and a local minimum at $(-3,-1)$.

7. For the cubic function defined by $f(x)=ax^3+bx^2+cx+d$, find the relationship between a,b,c such that:

a) there is no extrema

b) there are exactly two extrema

8. Find the values of a,b,c,d,e such that the quartic function $f(x)=ax^4+bx^3+cx^2+dx+e$ has turning points at $x=-1$, $x=1$, and $x=2$ and passes through $(-1,1)$ and $(1,0)$.

9. Find the critical points for $f(x)=x^n$, n is natural.

10. Find the critical numbers for $f(x)=[x]$ where $[x]$ is the largest integer less or equal to x.

11. Find the critical numbers for the Dirichlet function:

$$f(x)=\begin{cases}1 & if\ x\ is\ rational\\ 0 & if\ x\ is\ irrational\end{cases}$$

12. A function f has its derivative f' given by

$$f'(x)=x^3(x-1)^2(x+1)(x-2)$$

At what numbers x, if any, does f have a local minimum or maximum points.

13. Determine a and b such that the function $f(x)=\frac{ax}{x^2+b^2}$ has a local minimum at $x=-2$ and $f'(0)=1$.

Answers
Shortcuts: G=Global, L=Local, M=Maximum, m=minimum

A1. GM: $B(1,4)$; Gm: $E(4,0)$; LM: $B(1,4)$, $D(3,2)$; Lm: $C(2,1)$

2. a) GM: $B(2,3)$; Gm: none; LM: none; Lm: none
b) GM: none; Gm: none; LM: none; Lm: none
c) any point of the segment AB (endpoint A is not included, endpoint B is included) is GM or Gm; any interior point of the segment AB (endpoints A and B are not included) is LM or Lm
d) GM: $D(3,3)$, $F(4,3)$; Gm: $A(0,2)$, $C(2,2)$; LM: $D(3,3)$; Lm: $C(2,2)$

B1. a) $(-1,-1)$ b) $(0,0)$ or $(1,5)$ c) $(-1,-2)$ or $(-3,2)$ d) $(0,0)$ e) none f) $(0,0)$ g) none h) none i) $(-1,-1/2)$, $(1,1/2)$ j) $(0,1)$ k) $(0,1)$ l) none m) $(2,0)$ n) $(0,16)$ or $(\pm 2,0)$ o) $(-1,0)$ p) $(0,0)$ or $(1/4,-1/4)$ q) $(1,2)$ r) $(1,0)$ s) $(0,4)$ or $(\pm 4,0)$ t) $(-1,0)$, $(2,0)$ or $(1/2,-\sqrt[3]{9/4})$

2. a) $(2n\pi \pm \pi/2,\pm 1)$, n is integer b) none c) none d) none e) $(0,1)$ f) $(0,0)$ g) $(1/e,-1/e)$ h) $(-1,-1/e)$ i) $(0,1)$ j) $(\pm\sqrt{1/e},-1/e)$ k) $(0,0)$ or $(-2,4/e^2)$ l) $(0,0)$ or $(1,1/e^2)$ m) $(1,-1)$ n) $(0,1/2)$ o) $(0,1)$ p) $(\pm 1,1/e)$ or $(0,0)$ r) $(\sqrt{e},1/(2e))$

3. -8

4. $a=-3$ and $b=-120$

C1. a) $(x,1)$ is a local minimum and maximum where x is any real number b) none c) $(0,0)$ is a local minimum d) none e) none f) none g) none h) none i) none j) $(0,0)$ is a local minimum k) $(0,0)$ is a local minimum

2. a) none b) Lm: $(-1,0)$ or $(1,0)$; LM: $(0,1)$
c) LM: $(-1,-2)$; Lm: $(1,2)$ d) LM: $(1,1)$; Lm: $(0,0)$
e) LM: $(-1/3,32/27)$; Lm: $(1,0)$ f) none
g) Lm: $(-1,0)$ h) LM: $(1/2,-8)$
i) Lm: $(1/4,-1/4)$ j) Lm: $(16^{-1/3},256^{-1/3}-4^{-1/3})$
k) Lm: $(\pm 2,0)$, LM: $(0,4)$

3. a) $k=54$ b) $k=9$

D1. a) GM: $(-1,5)$, Gm: $(2,-1)$
b) GM: $(-2,8)$, Gm: $(1,-1)$
c) GM: $(-2,21)$, Gm: $(1,-6)$
d) GM: $(2,16)$, Gm: $(1,-1)$
e) GM: $(6,2)$, Gm: $(2,0)$
f) GM: $(3,5)$, Gm: $(-2,0)$
g) GM: $(4,4)$, Gm: $(0,0)$
h) GM: $(0,1)$, Gm: $(-2,-9/5)$
i) GM: $(8,2)$, Gm: $(-1,-1)$
j) GM: $(1,5)$ or $(4,5)$, Gm: $(2,4)$

2. a) GM: $(0,1)$ or $(2\pi,1)$, Gm: $(\pi,-1)$
b) GM: $(2,4)$, Gm: $(-2,1/4)$
c) GM: $(10,10)$, Gm: $(1/e,-1/(e\ln 10))$
d) GM: $(0,-1)$, Gm: $(2,2-e^2)$
e) GM: $(1,1/e)$, Gm: $(-1,-e)$
f) GM: $(-3,\ln 10)$, Gm: $(0,0)$
g) GM: $(4,e^4/16)$, Gm: $(2,e^2/4)$
h) GM: $(2\pi,2\pi)$, Gm: $(0,0)$

CQ1. a) $(0,0)$ or $(1/3,4/(9\sqrt[3]{9}))$ or $(1,0)$
b) $(-5,0)$ or $(19/7,(2916/49)(9\sqrt[3]{-9/7}))$ or $(4,0)$
c) $(\pm 2,0)$ or $\cong(2.17,-0.56)$ d) $(0,0)$, $(5/2,0)$ or $(15/7,-(225/49)\sqrt[3]{5/7})$ e) $(0,0)$ or $(2,-9\sqrt[3]{4})$
f) $(0,1)$ or $(\pm\sqrt{\sqrt{2}-1},2\sqrt{2}-2)$ g) $(1,-5)$, $(2,-4)$ or $(3,-5)$

2. a) Lm: $(\pm 2,0)$ b) LM: $(3/5,0)$; Lm: $(1,0)$
c) LM: $(1/2,27/16)$ d) LM: $(-2,-4)$; Lm: $(0,0)$
e) Lm: $(0,0)$ f) Lm: $(-2,0)$ or $(1,0)$; LM: $(-1/2,9/4)$
g) LM: $(3/4,3/(4\sqrt[3]{4}))$
h) LM: $(-4/5,16/(25\sqrt{5}))$; Lm: $(0,0)$
i) Lm: $(1/2,4/3)$ j) LM: $(-1,6)$; Lm: $(1,-6)$
k) Lm: $(-3/2,27/4)$
l) LM: $(0,0)$; Lm: $(2/11,-(729/1331)\sqrt[3]{4/121})$
m) LM: $(1/8,1/4)$ n) LM: $(-1,\sqrt[3]{2})$; Lm: $(1,-\sqrt[3]{2})$
o) Lm: $(-2,-4)$, LM: $(2,4)$

3. a) GM: $(-1,3)$, Gm: $(0,0)$ or $(2,0)$
b) GM: $(1,0)$ or $(2,0)$, Gm: $(0,-8)$
c) GM: $(8,64)$, Gm: $(6,-36)$
d) GM: $(3,4/5)$, Gm: $(0,1)$

4. $a=2$, $b=-4$, $c=0$, and $d=-6$

5. $a=10/27$, $b=5/9$, $c=-20/9$, and $d=-73/27$

6. $a=3/64$, $b=-21/32$, $c=9/8$, and $d=287/32$

7. a) $b^2 \le 3ac$ b) $b^2 > 3ac$

8. $a=-3/32$, $b=1/4$, $c=3/16$, $d=-3/4$, $e=13/32$

9. none if n is odd, $(0,0)$ if n is even

10. any number **11.** any number **12.** local minimum at $x=-1$ or $x=2$, local maximum at $x=0$ **13.** $a=4$ and $b=\pm 2$