1.3 Limits (II)

A. Piecewise-defined Functions

Let consider that $f(x)$ is a piecewise-defined function:

$$f(x)=\begin{cases} g(x), & x<a \\ c, & x=a \\ h(x), & x>a \end{cases}$$

Then:

$\lim_{x\to a^-} f(x)=\lim_{x\to a^-} g(x)$ and $\lim_{x\to a^+} f(x)=\lim_{x\to a^+} h(x)$

Ex: $f(x)=\begin{cases} 1-x^2, & x\le 0 \\ \sqrt{x+1}, & x>0 \end{cases}$. Find $\lim_{x\to 0} f(x)$.

B. Algebraic Identities

The following algebraic identities may be useful to find algebraically the limit of a function:

$a^2-b^2=(a-b)(a+b)$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$a^4-b^4=(a-b)(a^3+a^2b+ab^2+b^3)$

$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+...+b^{n-1})$

$a^n+b^n=(a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2-...b^{n-1}), n\ odd$

C. Rational Functions

Consider a rational function in the form:

$f(x)=\dfrac{P(x)}{Q(x)},\quad Q(x)\neq 0$ where $P(x)$ and $Q(x)$ are polynomial functions. If $x=a$ is a common zero of $P(x)$ and $Q(x)$ then the limit $\lim_{x\to a} f(x)$ leads to the indeterminative $\dfrac{0}{0}$. This indeterminative may be eliminated by dividing both $P(x)$ and $Q(x)$ by the common factor $x-a$.

Ex: Find $\lim_{x\to 1}\dfrac{x^2-1}{x-1}$.

D. Conjugate Radicals

In same cases, to eliminate an indeterminative of the form $\dfrac{0}{0}$ we multiply both the numerator and denominator by a conjugate radical in order to cancel out a common zero.

The conjugate radical of $\sqrt{a}-\sqrt{b}$ is $\sqrt{a}+\sqrt{b}$ and

$$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})=a-b$$

Ex: Find $\lim_{x\to 1}\dfrac{\sqrt{x}-1}{x-1}$.

E. Change of Variable

A change of the independent variable might simplify the process of evaluating limits.

By changing the independent variable we must change accordingly the number $x=a$ where we calculate the limit.

Ex: Find $\lim_{x\to 1}\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$.

F. Squeeze Theorem

Let assume that $g(x)\le f(x)\le h(x)$ on an open interval containing the number $x=a$ and $\lim_{x\to a} g(x)=\lim_{x\to a} h(x)=L$ .Then: $\lim_{x\to a} f(x)=L$

Ex: Find $\lim_{x\to 0} x\sin\left(\dfrac{1}{x}\right)$.

G. Fundamental Limits of Calculus

Calculus is based on 3 fundamental limits:

1. Power Functions (case $\dfrac{0}{0}$)

$$\lim_{x\to 0}\frac{(1+x)^{\alpha}-1}{x}=\alpha,\quad \alpha\in\mathbb{R}$$

Ex: Find $\lim_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}$.

2. Exponential Functions (case 1^{∞})

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$

Ex: Find $\lim_{x\to 0}(1+2x)^{\frac{3}{x}}$.

3. Trigonometric Functions (case $\dfrac{0}{0}$)

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Ex: $\lim_{x\to 0}\dfrac{\sin(2x)}{5x}$

A. Piecewise-defined Functions

1. Evaluate, if it exists.

a) $\lim_{x\to 1}|x-1|$ b) $\lim_{x\to 0}\frac{|x|}{x}$ c) $\lim_{x\to 1^+}\frac{|x-1|}{1-x}$

d) $\lim_{x\to 0}x|x|$ e) $\lim_{x\to 1}\frac{x^2-1}{|x-1|}$ f) $\lim_{x\to 0}\frac{|x|(x+1)}{x}$

2. The Heaviside function is defined by:

$$H(t)=\begin{cases}0 & \text{if } t<0\\ 1 & \text{if } t\ge 0\end{cases}$$

Evaluate, if it exits:

a) $\lim_{x\to 0^-}H(t)$ b) $\lim_{x\to 0^+}H(t)$ c) $\lim_{x\to 0}H(t)$

3. Consider the function:

$$f(x)=\begin{cases}x+1 & \text{if } x<-1\\ x^2 & \text{if } -1\le x\le 1\\ \sqrt{x} & \text{if } x>1\end{cases}$$

Find the following limits, if they exit.

a) $\lim_{x\to -1}f(t)$ b) $\lim_{x\to 0}f(t)$ c) $\lim_{x\to 1}f(t)$

4. Find c such that the function:

$$f(x)=\begin{cases}1+\sin x & \text{if } x<0\\ c-\cos x & \text{if } x\ge 0\end{cases}$$

has a limit as x approaches 0.

5. Find a and b such that the function:

$$f(x)=\begin{cases}a+e^x & \text{if } x<0\\ \sqrt{x} & \text{if } 0\le x<1\\ b+\ln x & \text{if } x\ge 1\end{cases}$$

has a limit everywhere (at any number).

6. The function $[x]$ is defined as the largest integer that is less than or equal to x. Find, if it exists:

a) $\lim_{x\to 0^-}[x]$ b) $\lim_{x\to 0^+}[x]$ c) $\lim_{x\to 0}[x]$

d) $\lim_{x\to 1}(x-[x])$ e) $\lim_{x\to 0}x[x]$ f) $\lim_{x\to 0}\frac{[x]}{x}$

g) $\lim_{x\to n^-}[x]$ *n is an integer*

h) $\lim_{x\to n^+}[x]$ *n is an integer*

7. The sign function is defined by:

$$\operatorname{sgn}(x)=\begin{cases}-1 & \text{if } x<0\\ 0 & \text{if } x=0\\ 1 & \text{if } x>1\end{cases}$$

Find, if it exists:

a) $\lim_{x\to 0^-}\operatorname{sgn}(x)$ b) $\lim_{x\to 0^+}\operatorname{sgn}(x)$ c) $\lim_{x\to 0}\operatorname{sgn}(x)$

d) $\lim_{x\to 0}\operatorname{sgn}(|x|)$ e) $\lim_{x\to 0}|\operatorname{sgn}(x)|$ f) $\lim_{x\to 0}x\operatorname{sgn}(x)$

B. Algebraic Identities

C. Rational Functions

1. Evaluate.

a) $\lim_{x\to -2}\frac{x^2-4}{x+2}$ b) $\lim_{x\to 1}\frac{x^2+x-2}{x-1}$

c) $\lim_{x\to 2}\frac{x^3-8}{x^2-4}$ d) $\lim_{x\to 3}\frac{x^2-2x-3}{x^2-4x+3}$

e) $\lim_{x\to -3}\frac{x^3+27}{x+3}$ f) $\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$

g) $\lim_{x\to 2}\frac{(x+2)^2-16}{x-2}$ h) $\lim_{x\to 1}\frac{x^4-1}{x^3-1}$

i) $\lim_{x\to 0}\frac{(3-x)^{-1}-(3+x)^{-1}}{x}$

D. Conjugate Radicals

1. Evaluate.

a) $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$ b) $\lim_{x\to 1}\frac{\sqrt{x}-x}{x-1}$

c) $\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$ d) $\lim_{t\to 0}\frac{\sqrt{3+t}-\sqrt{3}}{t}$

e) $\lim_{x\to 1}\frac{\frac{1}{\sqrt{x}}-1}{x-1}$ f) $\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)$

g) $\lim_{x\to 0}\frac{\sqrt{3-x}-\sqrt{3+x}}{x}$ h) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{2+x}-\sqrt{2-x}}$

i) $\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}$

E. Change of Variable

1. Evaluate.

a) $\lim_{x\to 8}\frac{x-8}{\sqrt[3]{x}-2}$ b) $\lim_{x\to 4}\frac{x^{3/2}-8}{x-4}$

F. Squeeze Theorem

1. Let $f(x)=\begin{cases}x^2 & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$. Find $\lim_{x\to 0}f(x)$.

G. Fundamental Limits of Calculus

1. Find the following limits:

a) $\lim_{x\to 0}\frac{\sqrt[3]{1+x}-1}{\sqrt[5]{1+2x}-1}$ b) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt[3]{1+x}}{\sqrt[4]{1+x}-\sqrt[5]{1+x}}$

2. Find the following limits:

a) $\lim_{x\to 0}\left(\frac{1+x}{1-x}\right)^{1/x}$ b) $\lim_{x\to 0}(1+x^2)^{\frac{1}{x}}$

c) $\lim_{x\to 0}(1+\sqrt{|x|})^{\frac{1}{x}}$ d) $\lim_{x\to 0}\frac{\ln(1+x)}{x}$

e) $\lim_{x\to 1}\frac{2\ln x}{x^2-1}$ f) $\lim_{x\to 0}\frac{\ln(1-3x)}{2x}$

3. Find the following limits:

a) $\lim_{x\to 0}\frac{1-\cos x}{x}$ b) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

c) $\lim_{x\to 0}\frac{\tan x}{x}$ d) $\lim_{x\to 0}\frac{\sin(x^2)}{\tan^2 x}$

CQ. Challenge Questions

1. The Dirichlet function is defined by

$$f(x)=\begin{cases}1 & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

Find, if it exists:

a) $\lim_{x\to 1} f(x)$ b) $\lim_{x\to \pi} f(x)$

2. Find a and b such that $\lim_{x\to 0}\frac{\sqrt{ax+b}-2}{x}=1$.

3. If the following limit exists $\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$ find the value of a and the value of the limit.

4. Evaluate $\lim_{x\to 0}\frac{\sqrt[3]{1+ax}-1}{x}$.

5. Find the limits if they exist ($[x]$ is the largest integer that is less than or equal to x).

a) $\lim_{x\to 0} x\left[\frac{1}{x}\right]$ b) $\lim_{x\to 0} x^2\left[\frac{1}{x}\right]$

6. Let $f(x)=\frac{1}{1+2^{1/x}}$. Determine whether $\lim_{x\to 0} f(x)$ exists.

7. Find the following limits:

a) $\lim_{x\to 0}\frac{x-\sin x}{x^3}$ b) $\lim_{x\to 0}(\cos x)^{\frac{1}{\sin x}}$

c) $\lim_{x\to 0}\frac{\ln\cos x}{x^2}$ d) $\lim_{x\to 0}\frac{(1+x)^\alpha-(1+x)^\beta}{(1+x)^\gamma-(1+x)^\delta}$

A1. a) 0 b) DNE c) -1 d) 0 e) DNE f) DNE
2. a) 0 b) 1 c) DNE
3. a) DNE b) 0 c) 1
4. 2
5. $a=-1$ and $b=1$
6. a) -1 b) 0 c) DNE d) DNE e) 0 f) DNE g) $n-1$ h) n
7. a) -1 b) 1 c) DNE d) 1 e) 1 f) 0
C1. a) -4 b) 3 c) 3 d) 2 e) 27 f) -1/4 g) 8 h) 4/3 i) 2/9
D1. a) 4 b) -1/2 c) 1/2 d) $1/(2\sqrt{3})$ e) -1/2 f) -1/2 g) $-1/\sqrt{3}$ h) $\sqrt{2}$ i) 3
E1. a) 12 b) 3
F1. 0
G1. a) 5/3 b) 10/3 c)
2. a) e^2 b) 1 c) DNE d) 1 e) 1 f) -3/2
3. a) 0 b) 1/2 c) 1 d) 1
CQ1. a) DNE b) DNE
2. $a=4$ and $b=4$
3. $a=15$ and $L=-1$
4. 1/3
5. a) 1 b) 0
6. DNE
7. a) 1/6 b) 1 c) 1/2 d) $\frac{\alpha-\beta}{\gamma-\delta}$

