1.2 Limits (I)

A. Overview

(*Limit*) The concepts of limit is related to the behaviour of a function $f(x)$ in the neighbourhood of a number $x = a$.

Consider a function $f(x)$ defined on a neighbourhood of the number $x = a$, eventually not defined at $x = a$.

B. Definitions

(*Left-Hand Limit*) If the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a with $x < a$, then

$$\lim_{x \to a^-} f(x) = L$$

Read: "the limit of the function $f(x)$ as x approaches a from the left equals L"

Ex:

(*Right-Hand Limit*) If the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a with $x > a$, then

$$\lim_{x \to a^+} f(x) = L$$

Read: "the limit of the function $f(x)$ as x approaches a from the right equals L"

Ex:

(*Limit*) If the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a, then

$$\lim_{x \to a} f(x) = L$$

Read: "the limit of the function $f(x)$ as x approaches a equals L"

Note that $\lim_{x \to a} f(x) = L$ is equivalent to:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

(The limit exists if both left-hand and right-hand limits exist and are equal. Otherwise the limit Does Not Exist (DNE)).

Ex:

C. Numerical Approach

(*Technology*) Technology (scientific or graphing calculators) can be used to analyse the behaviour of a function in the neighbourhood of a number $x = a$.

In order to do that, create a table of values like the following:

x approaches from left	$f(x)$	x approaches from right	$f(x)$
$a - 0.1$		$a + 0.1$	
$a - 0.01$		$a + 0.01$	
$a - 0.001$		$a + 0.001$	
…		…	
a		a	

Then analyse the trend (limit) of the values in the $f(x)$ columns to make estimations of the left-hand limit, right-hand limit and the limit of the function $f(x)$ as x approaches a.

Ex:

D. Elementary Functions

(*Basic Functions*) The following functions are called elementary:

a) constant function $f(x) = c, f : \mathbb{R} \to \{c\}$

b) power functions

- $f(x) = x^n, n \in \mathbb{N}, f : \mathbb{R} \to \mathbb{R}$
- $f(x) = \dfrac{1}{x^n}, n \in \mathbb{N}, f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$
- $f(x) = x^{\frac{p}{q}} = \sqrt[q]{x^p}$, p *and* $q \in \mathbb{I}, q > 0$. The domain depends on the sign of p and on the fact that q is odd or even.
- $f(x) = x^\alpha, \alpha \in \mathbb{R}, f : [0, \infty) \to [0, \infty)$

c) exponential function $f(x) = a^x, a > 0, a \neq 1$

d) logarithmic functions $f(x) = \log_a x, a > 0, a \neq 1$

e) trigonometric functions

- $\sin(x), \sin : \mathbb{R} \to [-1,1]$
- $\cos(x), \cos : \mathbb{R} \to [-1,1]$
- $\tan(x), \tan : \mathbb{R} \setminus (2n+1)\pi/2 \to \mathbb{R}$

f) absolute value function $f(x) = |x|, f : \mathbb{R} \to [0, \infty)$

(*Substitution*) In order to calculate the limit of a basic function at number a in the domain of the basic function use substitution:

$$\lim_{x \to a} f(x) = f(a)$$

Ex:

E. Limit Laws

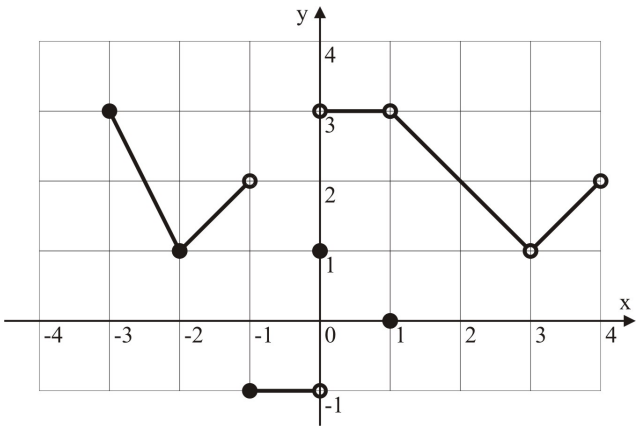
(*Limits Laws*) Functions can be combined arithmetic or by composition to create new functions. The following laws allow finding the limit for the new created functions:

1. $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a}[cf(x)] = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ *if* $\lim_{x \to a} g(x) \neq 0$
6. $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$, f *is a basic function*

Ex:

B. Definitions

For questions 1-3 use the function $f(x)$ defined by the following graph:



1. Find the limit of the function $f(x)$ as x approaches from the left to:

$a)-3 \quad b)-2 \quad c)-1 \quad d)\,0$

$e)\,1 \quad f)\,2 \quad g)\,3 \quad h)\,4$

2. Find the limit of the function $f(x)$ as x approaches from the right to:

$a)-3 \quad b)-2 \quad c)-1 \quad d)\,0$

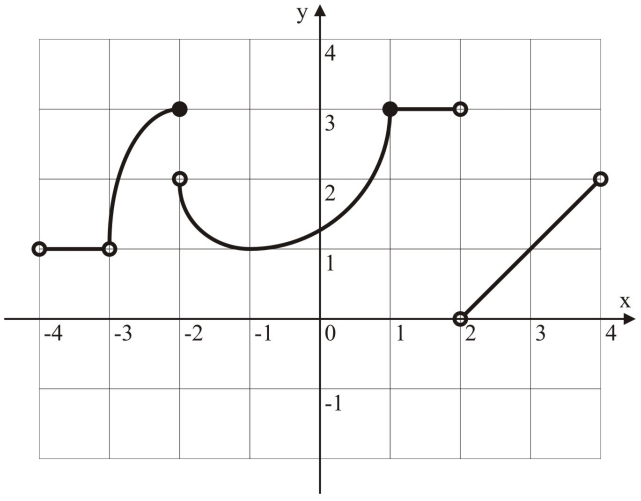
$e)\,1 \quad f)\,2 \quad g)\,3 \quad h)\,4$

3. Find the limit of the function $f(x)$ as x approaches:

$a)-3 \quad b)-2 \quad c)-1 \quad d)\,0$

$e)\,1 \quad f)\,2 \quad g)\,3 \quad h)\,4$

4. The function $f(x)$ defined by the following graph:



Find the limit of the function $f(x)$ as x approaches:

$a)-4 \quad b)-3 \quad c)-2 \quad d)-1$

$e)\,1 \quad f)\,2 \quad g)\,3 \quad h)\,4$

C. Numerical Approach

1. Use the numerical approach to estimate the following limits. Identify the indeterminate for each case.

a) $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$ b) $\lim\limits_{x\to 1^+}\dfrac{\sqrt{x}-1}{x-1}$ c) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}, x\ in\ °$

d) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$ e) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}$ f) $\lim\limits_{x\to 0}\dfrac{\ln(1+x)-1}{x}$

g) $\lim\limits_{x\to 0}\left(\dfrac{4^{|x|}+9^{|x|}}{2}\right)^{1/|x|}$ h) $\lim\limits_{x\to 0}|x|^x$ i) $\lim\limits_{x\to 0}\dfrac{3^x-2^x}{x}$

D. Elementary Functions

1. Evaluate the following limits.

a) $\lim\limits_{x\to -2} 3$ b) $\lim\limits_{x\to -3} x^2$ c) $\lim\limits_{x\to 4}\sqrt{x}$

d) $\lim\limits_{x\to -8}\sqrt[3]{x}$ e) $\lim\limits_{x\to -1} 2^x$ f) $\lim\limits_{x\to 4}\log_2 x$

g) $\lim\limits_{x\to \pi/6}\sin x$ h) $\lim\limits_{x\to -\pi/4}\cos x$ i) $\lim\limits_{x\to \pi/3}\tan x$

j) $\lim\limits_{x\to \pi/3}(\sec x-\csc x)$

E. Limit Laws

1. Evaluate the following limits.

a) $\lim\limits_{x\to -1}(x^3-2x^2+3x-1)$ b) $\lim\limits_{x\to 2}\dfrac{x+x^2}{x^3-2}$

c) $\lim\limits_{x\to 2}(3^x-x^3+\log_2 x^2)$ d) $\lim\limits_{x\to \pi/3}\dfrac{\sin x}{\cos x-1}$

e) $\lim\limits_{x\to \pi/2}\sqrt{\sin x}$ f) $\lim\limits_{x\to \pi} 2^{\cos x}$

g) $\lim\limits_{x\to 0}\cos\sqrt{x^2+x}$ h) $\lim\limits_{x\to -1}\ln\dfrac{2+x}{-x}$

CQ. Challenge Questions

1. Use the numerical approach to estimate the following limits. Identify the indeterminate for each case.

a) $\lim\limits_{x\to 0^+} x\ln x$ b) $\lim\limits_{x\to 0^+} x^x$

c) $\lim\limits_{x\to 0} e^{1/x}$ d) $\lim\limits_{x\to 0^+}(-\ln x)^x$

Answers

B1. a) DNE b) 1 c) 2 d) -1 e) 3 f) 2 g) 1 h) 2
2. a) 3 b) 1 c) -1 d) 3 e) 3 f) 2 g) 1 h) DNE
3. a) DNE b) 1 c) DNE d) DNE e) 3 f) 2 g) 1 h) DNE
4. a) DNE b) 1 c) DNE d) 1 e) 3 f) DNE g) 1 h) DNE
C1. a) 2 b) 1/2 c) 0.0174 d) 1/2 e) e f) 1 g) 6 h) 1 i) 0.405
D1. a) 3 b) 9 c) 2 d) -2 e) 1/2 f) 2 g) 1/2 h) $\sqrt{2}/2$ i) $\sqrt{3}$ j) $2-2/\sqrt{3}$
E1. a) -7 b) 1 c) 3 d) $-\sqrt{3}$ e) 1 f) 1/2 g) 1 h) 0
CQ1. a) 0 b) 1 c) DNE d) 1

