3.1 Increasing and Decreasing Functions

A. Increasing and Decreasing Functions

(*Informal Definitions*) A function is increasing if its graph is rising as you scan it from left to right. A function is decreasing if its graph is falling as you scan it from left to right.

(*Increasing Function*) A function f is increasing on the interval (a,b) if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in the interval (a,b).

(*Decreasing Function*) A function f is decreasing on the interval (a,b) if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in the interval (a,b).

(*Constant Function*) A function f is constant on the interval (a,b) if $f(x_1) = f(x_2)$ for every x_1 and x_2 in the interval (a,b).

(*Test for Intervals of Increase/Decrease*) Let f be a differentiable function on (a,b). Then:
If $f'(x) > 0$ for all $x \in (a,b)$, then f is increasing on (a,b) (on $[a,b]$ if f is continuous on $[a,b]$)
If $f'(x) < 0$ for all $x \in (a,b)$, then f is decreasing on (a,b) (on $[a,b]$ if f is continuous on $[a,b]$)
If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on (a,b) (on $[a,b]$ if f is continuous on $[a,b]$)

(*Sign Chart*) When sketching the graph of a function is useful to develop the sign chart for the function and its derivative as presented below.

x	$-\infty$	∞
$f(x)$		
$f'(x)$		
$\uparrow$ or $\downarrow$		

B. Challenge Theorems

(*Rolle's Theorem*) Suppose that the function f is continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = 0 = f(b)$, then there exists some number $c \in (a,b)$ such that $f'(c) = 0$.

(*Mean Value Theorem*) Suppose that the function f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists some number $c \in (a,b)$ such that $f(b) - f(a) = (b-a)f'(c)$.

Practice Questions

A. Increasing and Decreasing Functions

1. For each case, use the first derivative sign to find the intervals of increase or decrease.

a) $f(x) = 5$ b) $f(x) = x$ c) $f(x) = x^2$

d) $f(x) = x^3$ e) $f(x) = x^4$ f) $f(x) = \dfrac{1}{x}$

g) $f(x) = \dfrac{1}{x^2}$ h) $f(x) = \dfrac{1}{x^3}$ i) $f(x) = \sqrt{x}$

j) $f(x) = \sqrt[3]{x}$ k) $f(x) = \dfrac{1}{\sqrt{x}}$ l) $f(x) = \frac{1}{\sqrt[3]{x}}$

2. For each case, use the first derivative sign to find the intervals of increase or decrease.

a) $f(x) = \sin x$ b) $f(x) = \cos x$ c) $f(x) = \tan x$

d) $f(x) = e^x$ e) $f(x) = 10^x$ f) $f(x) = 0.5^x$

g) $f(x) = \ln x$ h) $f(x) = \log x$ i) $f(x) = \ln_{0.5} x$

3. For each case, use the first derivative sign to find the intervals of increase or decrease.

a) $f(x) = x^2 - 2x$ b) $f(x) = \dfrac{x}{x-1}$

c) $f(x) = \dfrac{x}{x^2+1}$ d) $f(x) = \sqrt{x}(x-1)$

e) $f(x) = x + \dfrac{1}{x}$ f) $f(x) = 2\sqrt{x-1} - x$

g) $f(x) = \dfrac{x^2-1}{x^2+1}$ h) $f(x) = \sqrt[3]{x^2-1}$

4. Find the intervals of increase or decrease.

a) $f(x) = x \ln x$ b) $f(x) = \dfrac{\ln x}{x}$

c) $f(x) = \dfrac{x}{\ln x}$ d) $f(x) = xe^x$

e) $f(x) = xe^{-x}$ f) $f(x) = x + \sin x$

g) $f(x) = x^2 + \cos x$ h) $f(x) = \ln x^2$

i) $f(x) = \dfrac{(x+1)^2}{e^x}$ j) $f(x) = x^2 \ln x$

Challenge Questions

1. For each case, use the first derivative sign to find the intervals of increase or decrease.

a) $f(x) = 2x^3 + 3x^2 - 12x + 1$ b) $f(x) = x^2(x+1)^2$

c) $f(x) = 3x^5 - 25x^3 + 60x - 5$ d) $f(x) = \dfrac{1}{x^2-4}$

e) $f(x) = 2\sqrt{x} - 3\sqrt[3]{x}$ f) $f(x) = \sqrt[3]{x-1}(x+1)^2$

g) $f(x) = x^3(x-1)^4$ h) $f(x) = |x^2 - 1|$

i) $f(x) = |x+1||x-2|$

2. Use the sign chart to sketch the graph for each function.

a) $f(x) = x^3 - 6x^2 + 9x + 2$

b) $f(x) = (3x^4 - 4x^3 - 12x^2 + 5)/5$

3. Where is the function $y = 12x^5 + 15x^4 - 20x^3 + 27$ decreasing?

4. Analyse the increase/decrease intervals for the functions:

a) $f(x) = \begin{cases} x^3, & x < 1 \\ (1/2)x + 2, & x \geq 1 \end{cases}$

b) $f(x) = \begin{cases} (1/2)x + 2, & x < 1 \\ x^3, & x \geq 1 \end{cases}$

Answers

A1. a) f is constant

b) increase on $(-\infty, \infty)$

c) decrease on $(-\infty, 0)$, increase on $(0, \infty)$

d) increase on $(-\infty, \infty)$

e) decrease on $(-\infty, 0)$, increase on $(0, \infty)$

f) decrease on $(-\infty, 0)$ or $(0, \infty)$

g) increase on $(-\infty, 0)$, decrease on $(0, \infty)$

h) decrease on $(-\infty, 0)$ or $(0, \infty)$

i) decrease on $(0, \infty)$

j) increase on $(-\infty, \infty)$

k) decrease on $(0, \infty)$

l) decrease on $(-\infty, 0)$ or $(0, \infty)$

2. a) increase on $(2n\pi - \pi/2, 2n\pi - \pi/2)$ and decrease on $(2n\pi + \pi/2, 2n\pi + 3\pi/2)$ where n is integer

b) increase on $(2n\pi - \pi, 2n\pi + \pi)$ and decrease on $(2n\pi, 2n\pi + \pi)$ where n is integer

c) increase on $(2n\pi - \pi/2, 2n\pi - \pi/2)$ or $(2n\pi + \pi/2, 2n\pi + 3\pi/2)$ where n is integer

d) increase on $(-\infty, \infty)$

e) increase on $(-\infty, \infty)$

f) decrease on $(-\infty, \infty)$

g) increase on $(0, \infty)$

h) increase on $(0, \infty)$

i) decrease on $(0, \infty)$

3. a) decrease on $(-\infty, 1)$, increase on $(1, \infty)$

b) decrease on $(-\infty, 1)$ or $(1, \infty)$

c) decrease on $(-\infty, -1)$ or $(1, \infty)$, increase on $(-1, 1)$

d) decrease on $(0, 1/3)$, increase on $(1/3, \infty)$

e)) increase on $(-\infty, -1)$ or $(1, \infty)$, decrease on $(-1, 0)$ or $(0, 1)$

f)) increase on $(1, 2)$, decrease on $(2, \infty)$

g) decrease on $(-\infty, 0)$, increase on $(0, \infty)$

h) decrease on $(-\infty, 0)$, increase on $(0, \infty)$

4. a) decrease on $(0, 1/e)$, increase on $(1/e, \infty)$

b) increase on $(0, e)$, decrease on (e, ∞)

c) increase on $(0, 1)$ or (e, ∞), decrease on $(1, e)$

d) decrease on $(-\infty, -1)$, increase on $(-1, \infty)$

e) increase on $(-\infty, 1)$, decrease on $(1, \infty)$

f) increase on $(-\infty, \infty)$

g) decrease on $(-\infty, 0)$, increase on $(0, \infty)$

h) decrease on $(-\infty, 0)$, increase on $(0, \infty)$

i) decrease on $(-\infty, -1)$ or $(1, \infty)$, increase on $(-1, 1)$

j) decrease on $(0, 1/\sqrt{e})$, increase on $(1/\sqrt{e}, \infty)$

CQ1. a) increase on $(-\infty, -2)$ or $(1, \infty)$, decrease on $(-2, 1)$

b) decrease on $(-\infty, -1)$ or $(-1/2, 0)$, increase on $(-1, -1/2)$ or $(0, \infty)$

c) increase on $(-\infty, -2)$ or $(-1, 1)$ or $(2, \infty)$, decrease on $(-2, -1)$ or $(1, 2)$

d) increase on $(-\infty, -2)$ or $(-2, 0)$ decrease on $(0, 1)$ or $(2, \infty)$

e) decrease on $(0, 1)$, increase on $(1, \infty)$

f) increase on $(-\infty, -1)$ or $(5/7, \infty)$, decrease on $(-1, 5/7)$

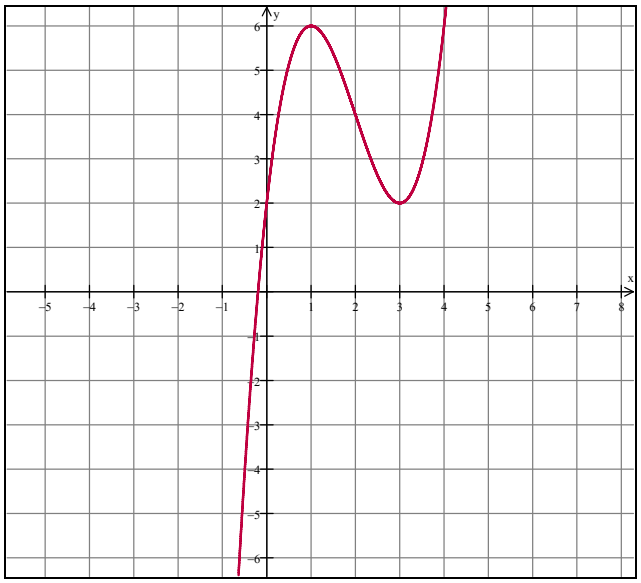
g) increase on $(-\infty, 3/7)$ or $(1, \infty)$, decrease on $(3/7, 1)$

h) decrease on $(-\infty, -1)$ or $(0, 1)$, increase on $(-1, 0)$ or $(1, \infty)$

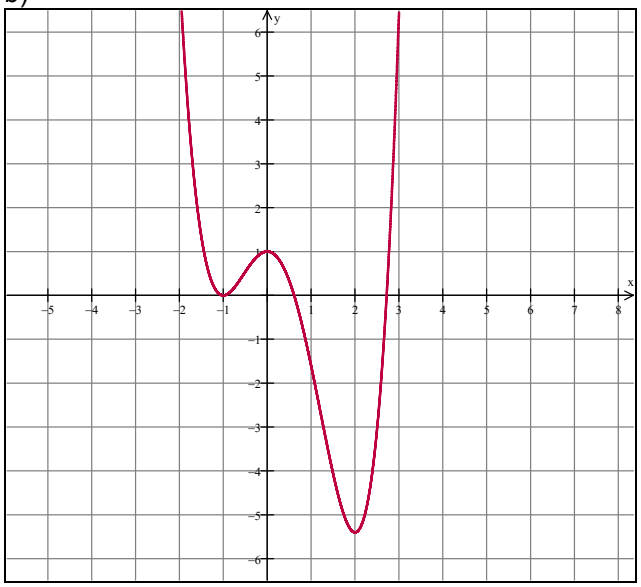
i) decrease on $(-\infty, -1)$ or $(1/2, 2)$, increase on $(-1, 1/2)$ or $(2, \infty)$

2. See the graphs below.

a)



b)



3. $\left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$

4. a) increase on $(-\infty, \infty)$

b) increase on $(-\infty, 1)$ or $(1, \infty)$

