2.6 Higher Derivatives

A. Higher Derivatives

(*Second Derivative*) The second derivative of a function $y = f(x)$ is defined as the derivative of the derivative f' of the function:

$$f'' = (f')'$$

Or:

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\frac{d}{dx} f(x)$$

Read: “f double prime”

(*Alternative Notations*) Other notations are used for the second derivative as:

$$y'' = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2 y}{dx^2}$$

(*Interpretation*) The second derivative $f''(x)$ represents the instantaneous rate of change of the slope of the tangent line to the curve $y = f(x)$.

(*Third Derivative*) The third derivative of a function $y = f(x)$ is defined as the derivative of the second derivative f'' of the function:

$$f''' = (f'')' = ((f')')'$$

Or:

$$f'''(x) = \frac{d}{dx} f''(x) = \frac{d}{dx}\frac{d}{dx} f'(x) = \frac{d}{dx}\frac{d}{dx}\frac{d}{dx} f(x)$$

Read: “f triple prime”.

(*Alternative Notations*) Other notations are used for the third derivative as:

$$y''' = \frac{d}{dx}\frac{d}{dx}\frac{dy}{dx} = \frac{d^3 y}{dx^3}$$

(*Fourth Derivative*) For the fourth derivative of the function $y = f(x)$ we have:

$$f^{(4)}(x) = \frac{d}{dx} f'''(x)$$

or

$$y^{(4)} = \frac{d}{dx}\frac{d}{dx}\frac{d}{dx}\frac{dy}{dx} = \frac{d^4 y}{dx^4}$$

(*nth Derivative*) Similarly, the nth derivative of the function $y = f(x)$ is defined by:

$$f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x)$$

or

$$y^{(n)} = \underbrace{\frac{d}{dx}\frac{d}{dx}\cdots\frac{dy}{dx}}_{n\ times} = \frac{d^n y}{dx^n}$$

(*Factorial Functions*) Factorial functions are defined by:

$$n! = n(n-1)(n-2)...(2)(1), \quad 0! = 1$$

$$n!! = n(n-2)(n-4)...(2)\ or\ (1), \quad 0!! = 1, \quad 1!! = 1$$

Practice Questions

A. Higher Derivatives

1. For each case, find the second derivative.

a) $y = 1$ b) $y = 1 + 2x$

c) $y = 1 - 3x + 4x^2$ d) $y = 4 - 2x + x^2 - 2x^3$

e) $y = \frac{1}{x}$ f) $y = \frac{1}{x^2}$

g) $y = \frac{1}{x^5}$ h) $y = \sqrt{x}$

i) $y = \sqrt[3]{x}$ j) $y = \frac{x}{x+1}$

k) $y = \sqrt{x^2 + 1}$ l) $y = x\sqrt{x}$

m) $y = x\sqrt{x+1}$ n) $y = \frac{1}{x^2 + 1}$

2. For each case, find the second derivative.

a) $y = \sin x$ b) $y = \cos x$

c) $y = e^x$ d) $y = \ln x$

e) $y = 2^x$ f) $y = \log x$

g) $y = \log_3 x$ h) $y = \tan x$

3. For each case, find the third derivative.

a) $y = x^4 - 2x^3$ b) $y = \sqrt{x}$

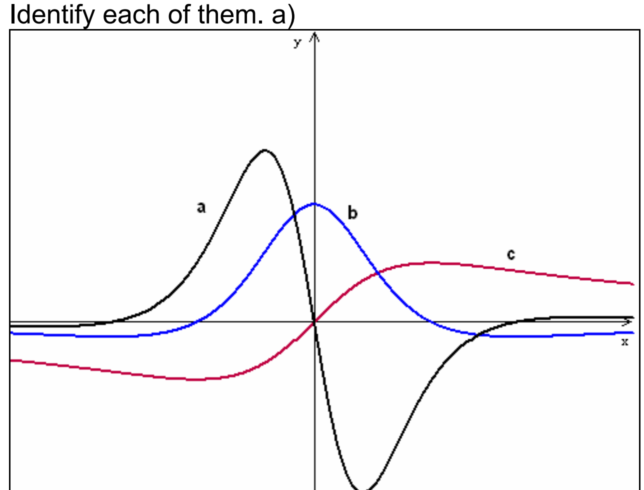
c) $y = \frac{1}{x}$ d) $y = \frac{1}{\sqrt{x}}$

e) $y = \sqrt[3]{x}$ f) $y = \frac{1}{\sqrt[3]{x}}$

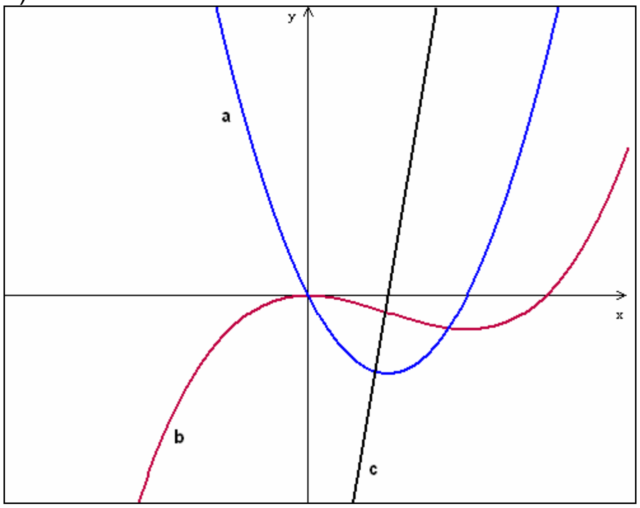
4. Find the third derivative of $y = \frac{1}{x^2 + x}$.

5. Find the first, second and the third derivative of $= \sqrt{x^2 - 2x}$.

6. The graphs of a function and its first and second derivatives are represented on the same grid. Identify each of them. a)



b)



7. Show that $y = x^3 + 3x + 1$ satisfies $y''' + xy'' = 2y'$.

Challenge Questions

1. For each case, find the second derivative.

a) $y = x^2 \sin x$ b) $y = x^2 \ln x$

c) $y = \dfrac{e^x}{x}$ d) $y = \dfrac{\ln x}{e^x}$

2. For each case, find the third derivative.

a) $y = \dfrac{1-x}{1+x}$ b) $y = \sqrt{5x-1}$

3. For each case, find the nth derivative.

a) $y = \dfrac{1}{x}$ b) $y = \dfrac{1}{x^2}$

c) $y = \sqrt{x}$ d) $y = \dfrac{1}{x^2 + x}$

e) $y = \dfrac{1}{x^2 - 4}$

4. For each case, find the nth derivative.

a) $y = \sin x$ b) $y = \cos x$

c) $y = e^x$ d) $y = 2^x$

e) $y = \ln x$ f) $y = \log_2 x$

g) $y = x \ln x$ h) $y = xe^x$

i) $y = xe^{-x}$

5. Let $(fg)(x) = f(x)g(x)$. The first derivative of the function fg is given by the product rule:

$$(fg)' = f'g + fg'$$

Find similar expressions for:

a) the second derivative $(fg)''$

b) the third derivative $(fg)'''$

c) the nth derivative $(fg)^{(n)}$

6. If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable, find a formula for $\dfrac{d^2y}{dx^2}$.

Answers

A1. a) $y'' = 0$ b) $y'' = 0$ c) $y'' = 8$ d) $y'' = 2 - 12x$

e) $y'' = \dfrac{2}{x^3}$ f) $y'' = \dfrac{6}{x^4}$ g) $y'' = \dfrac{30}{x^7}$ h) $y'' = \dfrac{-1}{4\sqrt[3]{x^2}}$

i) $y'' = \dfrac{-2}{9x\sqrt[3]{x^2}}$ j) $y'' = \dfrac{-2}{(x+1)^3}$ k) $y'' = \dfrac{1}{(x^2+1)^{3/2}}$

l) $y'' = \dfrac{3}{4\sqrt{x}}$ m) $y'' = \dfrac{3x+4}{4(x+1)^{3/2}}$ n) $y'' = \dfrac{2(3x^2-1)}{(x^2+1)^3}$

2. a) $y'' = -\sin x$ b) $y'' = -\cos x$ c) $y'' = e^x$

d) $y'' = -1/x^2$ e) $y'' = (\ln 2)^2(2^x)$ f) $y'' = \dfrac{-1}{(\ln 10)x^2}$

g) $y'' = \dfrac{-1}{(\ln 3)x^2}$ h) $y'' = \dfrac{2\sin x}{\cos^3 x}$

3. a) $y''' = 24x - 12$ b) $y''' = (3/8)x^{-5/2}$ c) $y''' = -6/x^4$

d) $y''' = (-15/8)x^{-7/2}$ e) $y''' = (10/27)x^{-8/3}$

f) $y''' = (-28/27)x^{-10/3}$

4. $y''' = \dfrac{-6}{x^4} + \dfrac{6}{(x+1)^4}$

5. $y' = \dfrac{x-1}{\sqrt{x^2-2x}}$, $y'' = \dfrac{-1}{x^2-2x}$, $y''' = \dfrac{2(x-1)}{(x^2-2x)^2}$

6. a) f is c, f' is b, f'' is a

b) f is b, f' is a, f'' is c

CQ1. a) $y'' = 2\sin x + 4x\cos x - x^2 \sin x$

b) $y'' = 2\ln x + 1$ c) $y'' = \left(\dfrac{1}{x} - \dfrac{2}{x^2} + \dfrac{2}{x^3}\right)e^x$

d) $y'' = \left(\ln x - \dfrac{2}{x} - \dfrac{1}{x^2}\right)e^{-x}$

2. a) $y''' = -12(1+x)^{-4}$ b) $y''' = (375/8)(5x-1)^{-5/2}$

3. a) $y^{(n)} = \dfrac{(-1)^n n!}{x^{n+1}}$ b) $y^{(n)} = \dfrac{(-1)^n n!}{x^{n+2}}$

c) $y^{(n)} = \dfrac{(-1)^{n-1}(2n-3)!!}{2^n x^{(2n-1)/2}}$

d) $y^{(n)} = (-1)^n n!\left(\dfrac{1}{x^{n+1}} - \dfrac{1}{(x+1)^{n+1}}\right)$

e) $y^{(n)} = \dfrac{(-1)^n n!}{4}\left(\dfrac{1}{(x-2)^{n+1}} - \dfrac{1}{(x+2)^{n+1}}\right)$

4. a) $y^{(2n)} = (-1)^n \sin x$, $y^{(2n+1)} = (-1)^n \cos x$

b) $y^{(2n)} = (-1)^n \cos x$, $y^{(2n+1)} = (-1)^{n+1} \sin x$

c) $y^{(n)} = e^x$ d) $y^{(n)} = (\ln 2)^n (2^x)$

e) $y^{(n)} = (-1)^{n-1}(n-1)!x^{-n}$

f) $y^{(n)} = \dfrac{(-1)^{n-1}(n-1)!}{(\ln 2)^n x^n}$

g) $y' = \ln x + 1$, $y^{(n)} = \dfrac{(-1)^n (n-2)!}{x^{n-1}}$ $if\ n \geq 2$

h) $y^{(n)} = (n + x)e^x$

i) $y^{(n)} = (-1)^{n-1}(n - x)e^{-x}$

5. a) $(fg)'' = f''g + 2f'g' + fg''$

b) $(fg)''' = f'''g + 3f''g' + 3f'g'' + fg'''$ c)

$(fg)^{(n)} = C(n,0)f^{(n)}g + C(n,1)f^{(n-1)}g^{(1)} + \ldots +$

$+ C(n,k)f^{(n-k)}g^{(k)} + \ldots + C(n,n-1)f^{(1)}g^{(n-1)} + C(n,n)fg^{(n)}$

where $C(n,k) = \dfrac{n!}{k!(n-k)!}$

6. $\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{du^2}\left(\dfrac{du}{dx}\right)^2 + \dfrac{dy}{du}\dfrac{d^2u}{dx^2}$

