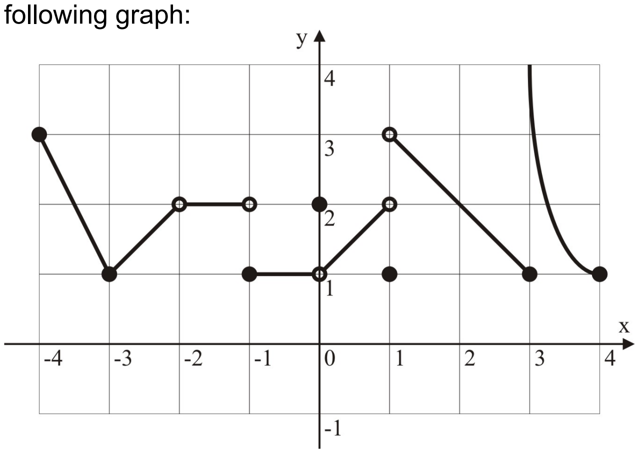
1.4 Continuity

A. Overview

(*Continuity*) The concept of limit continuity is related to the behaviour of a function $f(x)$ in the neighbourhood of a number $x = a$ including this number.

Example. Consider the function $f(x)$ defined by the following graph:



B. Continuity

> (*Continuity from the Left*) A function $f(x)$ is continuous from the left at a number a if:
> $$\lim_{x \to a^-} f(x) = f(a)$$

Example. Analyse if $f(x)$ is continuous from the left at:

$a) -4$ $b) -3$ $c) -2$ $d) -1$ $e) 0$
$f) 1$ $g) 2$ $h) 3$ $i) 4$

> (*Continuity from the Right*) A function $f(x)$ is continuous from the right at a number a if:
> $$\lim_{x \to a^+} f(x) = f(a)$$

Example. Analyse if $f(x)$ is continuous from the right at:

$a) -4$ $b) -3$ $c) -2$ $d) -1$ $e) 0$
$f) 1$ $g) 2$ $h) 3$ $i) 4$

> (*Continuity*) A function $f(x)$ is continuous at a number a if:
> $$\lim_{x \to a} f(x) = f(a)$$

Note: A function $f(x)$ is continuous at a if the following 3 conditions are met:

a) $f(a)$ exists (the function is defined at a)
b) $\lim_{x \to a} f(x)$ exists
c) $f(a)$ and $\lim_{x \to a} f(x)$ are equal

Example. Analyse if $f(x)$ is continuous at:

$a) -4$ $b) -3$ $c) -2$ $d) -1$ $e) 0$
$f) 1$ $g) 2$ $h) 3$ $i) 4$

C. Discontinuity

> (*Discontinuity*) If a function $f(x)$ *is not continuous* at a then $f(x)$ is called discontinuous at a or $f(x)$ has a discontinuity at a .

Example. Find the points of discontinuity for the function $f(x)$.

> (*Removable Discontinuity*) If $\lim_{x \to a} f(x)$ exists but $f(a)$ does not exist or is not equal to $\lim_{x \to a} f(x)$, then the discontinuity at a is called removable because it can be removed by redefining the function at a as $f(a) = \lim_{x \to a} f(x)$.

Example. Find the points of removable discontinuity for the function $f(x)$.

> (*Jump Discontinuity*) If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ then the discontinuity at a is called jump discontinuity because the function "jumps" from one value to another.

Example. Find the points of jump discontinuity for the function $f(x)$.

> (*Infinite Discontinuity*) If at least one of $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ is unbounded (approaches $+\infty$ or $-\infty$) then the discontinuity at a is called infinite discontinuity

Example. Find the points of infinite discontinuity for the function $f(x)$.

D. Continuity over an Interval

> (*Continuity over an Interval*) A function is continuous over an open interval if it is continuous at every number in that interval.
> A function is continuous over $[a,b]$ if:
> a) the function is continuous over (a,b)
> b) the function is continuous from the right at a
> c) the function is continuous from the left at b

(*Elementary Functions*) Elementary functions are continuous on their domain. (⇨D1)

(*Algebraic Combinations*) If f and g are continuous at a, then $f+g$, $f-g$, fg, f/g where $g(a)\neq 0$ are also continuous at a. These algebraic combinations are continuous on $D_f \cap D_g$ ($g(x)\neq 0$ for f/g).

(⇨D2)

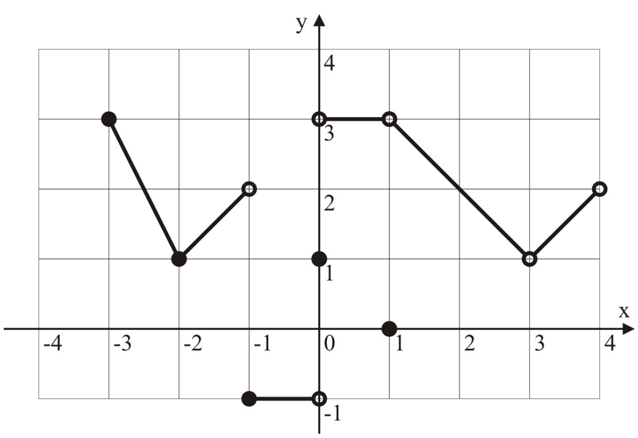
(*Composition of Functions*) If g is continuous at a and f is continuous at $g(a)$ then $f \circ g$ is continuous at a.

(*Intervals of Continuity*) A discontinuity splits the graph of a function in *branches*. A branch can be drawn without lifting the pen from the paper. Each branch corresponds to an *interval of continuity* of the function.
Example. List the intervals of continuity for the function $f(x)$.

Practice Questions

BC. Continuity and Discontinuity

1. The function $f(x)$ is defined by the following graph:



Analyse the continuity of the function $f(x)$ at:

$a)-3$ $b)-2$ $c)-1$ $d)\,0$

$e)\,1$ $f)\,2$ $g)\,3$ $h)\,4$

2. Classify each discontinuity as removable, jump, or infinite discontinuity for the function defined at example 1.

3. The Heaviside function is defined by:

$$H(t)=\begin{cases}0 & if\ t<0\\ 1 & if\ t\geq 0\end{cases}$$

Analyse the continuity of this function.

4. The function $[x]$ is defined as the largest integer that is less than or equal to x. Analyse the continuity of this function.

5. Analyse the continuity of the sign function:

$$\operatorname{sgn}(x)=\begin{cases}-1 & if\ x<0\\ 0 & if\ x=0\\ 1 & if\ x>1\end{cases}$$

6. Analyse the continuity for following functions:

a) $f(x)=\dfrac{x-2}{x^2-x-2}$ b) $f(x)=\dfrac{|x|}{x}$

c) $f(x)=1-\sqrt{1-x^2}$ d) $f(x)=\dfrac{3-\sqrt{x}}{9-x}$

7. Consider the function:

$$f(x)=\begin{cases}1-|x| & if\ |x|\leq 1\\ |x|-1 & if\ 1<|x|\leq 2\\ (x-3)^2 & if\ x>2\\ (x+3)^2 & if\ x<-2\end{cases}$$

Determine any value of x at which f is discontinuous.

8. Let

$$f(x)=\begin{cases}x & if\ x<0\\ x^2 & if\ 0\leq x\leq 1\\ x^3 & if\ x>1\end{cases}.$$

Show that f is continuous on $(-\infty,\infty)$.

9. Determine the discontinuities of the following function:

$$f(x)=\begin{cases}x^3 & if\ x\leq -1\\ x^2-2 & if\ -1<x<0\\ 3-x & if\ 0\leq x<2\\ \dfrac{4x-1}{x-1} & if\ 2\leq x<4\\ \dfrac{15}{7-x} & if\ 4<x<7\\ 5x+2 & if\ 7\leq x\end{cases}$$

10. Redefine the following functions to remove the discontinuity.

a) $f(x)=\dfrac{x^2-1}{x-1}$ b) $f(x)=\dfrac{x^2-3x+2}{x-2}$

c) $f(x)=\dfrac{9-x^2}{4-\sqrt{7+x^2}}$ d) $f(x)=\dfrac{\sqrt{x^2+4}-2}{x^2}$

e) $f(x)=\dfrac{\sqrt{x+4}-3}{\sqrt{x-5}}$

11. For what value of the constant c is the function

$$f(x)=\begin{cases}x+c & if\ x<2\\ cx^2+1 & if\ x\geq 2\end{cases}$$

continuous at every number?

12. For what value of k is the following function continuous?

$$f(x)=\begin{cases}\dfrac{\sqrt{7x+2}-\sqrt{6x+4}}{x-2} & \text{if } x\ge -2/7 \text{ and } x\ne 2\\ k & \text{if } x=2\end{cases}$$

13. For what value of a is the following function continuous at every number?

$$f(x)=\begin{cases}a^2x^2 & \text{if } x\in(-\infty,2]\\ (1-a)x & \text{if } x\in(2,\infty)\end{cases}$$

14. Let

$$f(x)=\begin{cases}4x & \text{if } x\le -1\\ ax+b & \text{if } -1<x<2\\ -5x & \text{if } x\ge 2\end{cases}$$

Find the values of a and b such that the function f is continuous on $(-\infty,\infty)$.

15. Give necessary and sufficient conditions on a and b for the function

$$f(x)=\begin{cases}ax-b & \text{if } x\le 1\\ 3x & \text{if } 1<x<2\\ bx^2-a & \text{if } x\ge 2\end{cases}$$

to be continuous at $x=1$ but discontinuous at $x=2$.

D. Continuity over an Interval

1. For each case specify the interval(s) where the function is continuous.

a) $f(x)=x^2$ b) $f(x)=\dfrac{1}{x^2-1}$

c) $f(x)=\sqrt{x-2}$ d) $f(x)=\ln(x+2)$

e) $f(x)=2^x$ f) $f(x)=\tan x$

2. For each case specify the interval(s) where the function is continuous.

a) $f(x)=x+\dfrac{1}{x}$ b) $f(x)=x^3+\sqrt{x}$

c) $f(x)=\sec x+\csc x$ d) $f(x)=\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

3. List the intervals of continuity for the function defined at example B1.

CQ. Challenge Questions

1. Analyse the continuity of the Dirichlet function:

$$f(x)=\begin{cases}1 & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

2. Redefine the following functions to remove the discontinuity.

a) $f(x)=\dfrac{\sin x}{x}$ b) $f(x)=\dfrac{1-\cos x}{x^2}$

Answers

BC1. a) continuous from the right b) continuous c) discontinuous d) discontinuous e) discontinuous f) continuous g) discontinuous h) discontinuous

2. removable discontinuity at $1,3$ and 4; jump discontinuity at -1 and 0

3. discontinuous at 0, continuous on $(-\infty,0)\cup(0,\infty)$

4. discontinuous at any integer, continuous on any interval $(n,n+1)$ where n is an integer

5. discontinuous at 0, continuous on $(-\infty,0)\cup(0,\infty)$

6. a) discontinuous at 2 (removable) b) discontinuous at 0 (jump) c) continuous on $[-1,1]$ d) continuous on $[0,9)$ or $(9,\infty)$; discontinuous at 9 (jump)

7. continuous everywhere

9. discontinuous at 0 (jump), 2 (jump), 4 (not defined at 4 and infinite from the right), and 7 (infinite from the left)

10. a) $g(x)=\begin{cases}f(x) & \text{if } x\ne 1\\ 2 & \text{if } x=1\end{cases}$ b) $g(x)=\begin{cases}f(x) & \text{if } x\ne 2\\ 1 & \text{if } x=2\end{cases}$

c) $g(x)=\begin{cases}f(x) & \text{if } x\ne \pm 3\\ 8 & \text{if } x=\pm 3\end{cases}$ d) $g(x)=\begin{cases}f(x) & \text{if } x\ne 0\\ 1/4 & \text{if } x=0\end{cases}$

e) $g(x)=\begin{cases}f(x) & \text{if } x>5\\ 0 & \text{if } x=5\end{cases}$

11. $c=1/3$

12. $k=1/8$

13. $a=-1$ or $a=1/2$

14. $a=-2$ and $b=-6$

15. $a-b=3$ and $6\ne 4b-a$

D1. a) continuous at any number b) continuous on $(-\infty,-1)$, $(-1,1)$ or $(1,\infty)$ c) continuous on $[2,\infty)$ d) continuous on $(-2,\infty)$ e) continuous at any number f) continuous on any interval $(n\pi-\pi/2, n\pi+\pi/2)$ where n is an integer

2. a) continuous on $(-\infty,0)$ or $(0,\infty)$ b) continuous on $[0,\infty)$ c) continuous on any interval $(n\pi/2,(n+1)\pi/2)$ where n is an integer d) continuous on $(-\infty,0)$ or $(0,\infty)$

3. continuous on $[-3,-1)$, $(-1,0)$, $(0,1)$, $(1,3)$ or $(3,4)$

CQ1. discontinuous at any number

2. a) $g(x)=\begin{cases}f(x) & \text{if } x\ne 0\\ 1 & \text{if } x=0\end{cases}$ b) $g(x)=\begin{cases}f(x) & \text{if } x\ne 0\\ 1/2 & \text{if } x=0\end{cases}$

