3.3 Concavity and Points of Inflection

A. Concavity

(*Concavity Upward*) A graph of a function f is called *concave upward* on an open interval (a,b) if the graph lies above of all its tangents on the interval (except for the point of tangency).
The tangent line rotates counter-clockwise and its slope increases as the point of tangency runs from the left to the right along the curve.
Figure.

(*Concavity Downward*) A graph of a function f is called *concave downward* on an open interval (a,b) if the graph lies below of all its tangents on the interval (except for the point of tangency).
The tangent line rotates clockwise and its slope decreases as the point of tangency runs from the left to the right along the curve.
Figure.

(*Test for Concavity*) Suppose that the function f is twice differentiable on an open interval (a,b).
(a) If $f''(x)>0$ for all $x\in(a,b)$, then the graph of f is concave upward on (a,b).
(b) If $f''(x)<0$ for all $x\in(a,b)$, then the graph of f is concave downward on (a,b).

(*Point of Inflection*) A point P is called a *point of inflection* if the concavity of the graph changes at P (from concave upward to concave downward or from concave downward to concave upward).
Figure.

(*Finding Inflection Points*) If $P(c,f(c))$ is an inflection point on the graph of the function f, then $f''(c)=0$ or $f''(c)$ does not exist.
Figure.

Example 1. For each case, find the intervals on which the graph is concave upward or downward. State any inflection points and sketch the graph.
a) $y=x^3-3x+1$
b) $y=x^4-2x^3$

B. Maximum and Minimum Values

(*Second Derivative Test*) Suppose that the function f is twice differentiable on an open interval that contains the number c and $f'(c)=0$.
(a) If $f''(c)>0$, then f has a local minimum at c.
(a) If $f''(c)<0$, then f has a local maximum at c.
Figure.

Example 2. Use the second derivative test to find the local minimum and maximum values for $y=\dfrac{x}{x^2+1}$.

(*Inconclusive Case*) If $f'(c)=0$ and $f''(c)=0$, then f may have a minimum, a maximum or neither. To conclude, use the first derivative test.
Figure.

Example 3. Find the local minimum and maximum values for:
a) $y=x^3$
b) $y=x^4$

Example 4. Consider $y=x^{1/3}(x+3)^{2/3}$.
Find the intervals on which the graph is concave upward or downward. State any inflection points and sketch the graph.

Practice Questions

A. Concavity

1. For each case, find the intervals of concavity and the points of inflection.
a) $f(x)=x^2-4x+3$ b) $f(x)=x^3-x$
c) $f(x)=x^4-6x^2$
d) $f(x)=x^4-3x^3+3x^2-x$
e) $f(x)=\dfrac{1}{4}x^4-x^3+4x+2$
f) $f(x)=(x^2-1)^3$ g) $f(x)=x+\dfrac{1}{x}$
h) $f(x)=x^2+\dfrac{1}{x}$ i) $f(x)=x+\dfrac{1}{x^2}$
j) $f(x)=x\sqrt{x^2+1}$ k) $f(x)=\dfrac{x}{x^2-1}$
l) $f(x)=\dfrac{x+2}{x-2}$ m) $f(x)=x^{1/3}$
n) $f(x)=x^{2/3}$ o) $f(x)=(x+1)^{5/3}$
p) $f(x)=(1-x)^2(1+x)^2$
q) $f(x)=(x-1)(x+1)^3$
r) $f(x)=\dfrac{x}{(x+1)^2}$ s) $f(x)=\dfrac{1}{x^2-1}$
t) $f(x)=\dfrac{1}{x^2+1}$ u) $f(x)=\dfrac{1-x^2}{x^3}$

2. For each case, find the intervals of concavity and the points of inflection.

a) $f(x)=\begin{cases} x^3, & x<1 \\ 3x-2, & x\geq 1 \end{cases}$

b) $f(x)=\begin{cases} 2x+4, & x\leq -1 \\ 3-x^2, & x>-1 \end{cases}$

3. For each case, find the intervals of concavity and the points of inflection.

a) $f(x)=xe^x$ b) $f(x)=xe^{-x}$

c) $f(x)=x^2e^x$ d) $f(x)=x\ln x$

e) $f(x)=x^2\ln x$ f) $f(x)=x+\cos x$

g) $f(x)=xe^{x^2}$ h) $f(x)=\ln(1+x^2)$

4. Sketch the graph of a function that satisfies all of the given conditions.

a) $f'(x)>0$ and $f''(x)<0$

b)
$f'(-1)=f'(1)=0,\ f'(x)<0\ \mathit{if}\ |x|<1,\ f'(x)>0\ \mathit{if}\ |x|>1,$
$f(-1)=4,\ f(1)=0,\ f''(x)<0\ \mathit{if}\ x<0,\ f''(x)>0\ \mathit{if}\ x>0$

c)
$f'(-1)=0,\ f'(1)\ DNE,\ f'(x)<0\ \mathit{if}\ |x|<1,\ f'(x)>0$
$\mathit{if}\ |x|>1,\ f(-1)=4,\ f(1)=0,\ f''(x)<0\ \mathit{if}\ x\neq 1$

d)
$f'(-1)=0,\ f'(2)=0,\ f(-1)=f(2)=-1,\ f(-3)=4,$
$f'(x)=0\ \mathit{if}\ x<-3,\ f'(x)<0\ \mathit{on}\ (-3,-1)\ \mathit{and}\ (0,2),$
$f'(x)>0\ \mathit{on}\ (-1,0)\ \mathit{and}\ (2,\infty),\ f''(x)>0\ \mathit{on}\ (-3,0)\ \mathit{and}\ (0,5),$
$f''(x)<0\ \mathit{on}\ (5,\infty)$

e)
$f(0)=0,\ f(-1)=1,\ f'(-1)=0,\ f''(x)>0\ \mathit{on}\ (-\infty,-1),$
$f''(x)<0\ \mathit{on}\ (-1,0)\ \mathit{and}\ (0,\infty),\ f'(x)>0\ \mathit{for}\ x>0$

B. Maximum and Minimum Values

1. Use the second derivative test to find the local maximum and minimum values of each function.

a) $f(x)=x^2-2x$ b) $f(x)=x^3-6x^2$

c) $f(x)=x^3-3x+2$ d) $f(x)=x^4-6x^2-5$

e) $f(x)=x^4-8x^3$ f) $f(x)=x+\dfrac{1}{x}$

g) $f(x)=\dfrac{x}{x^2+1}$ h) $f(x)=\dfrac{x}{(x-1)^2}$

i) $f(x)=(1-x^2)^2-2$ j) $f(x)=x+3x^{2/3}$

k) $f(x)=\dfrac{1}{4}x^4-x^3-2x^2+12x+5$

Challenge Questions

1. For each case, find the intervals of concavity and the points of inflection.

a) $f(x)=\dfrac{x^2}{\sqrt{1+x}}$ b) $f(x)=\sqrt[3]{x}(x+4)$

c) $f(x)=x^{2/3}(x+5)$ d) $f(x)=x\sqrt{4-x^2}$

e) $f(x)=3x^{5/3}-5x$ f) $f(x)=x^{2/3}(6-x)^{1/3}$

g) $f(x)=(x^2-1)^{2/3}$ h) $f(x)=x^{1/3}+x^{2/3}$

i) $f(x)=x^2(x-1)^{1/3}$ j) $f(x)=x\sqrt[3]{3x+2}$

k) $f(x)=\dfrac{3(x^2-1)}{x^2+3}$

l) $f(x)=(x-1)^{1/3}+(x+1)^{1/3}$

m) $f(x)=(x-1)^{2/3}-(x+1)^{2/3}$

n) $f(x)=4x^{1/3}-x^{4/3}$

2. Show that the function $f(x)=x|x|$ has an inflection point at $(0,0)$ but $f''(0)$ does not exist.

3. For what values of c does the polynomial $P(x)=x^4+cx^3+x^2$ have
a) two inflection points
b) one inflection point
c) no inflection point

4. Suppose that f'' is continuous and $f'(c)=f''(c)=0$, but $f'''(c)>0$.
a) Does f have a local maximum or minimum at c?
b) Does f have an inflection point at c?

5. Find d given that $(d,f(d))$ is a point of inflection of the graph of $f(x)=(x-a)(x-b)(x-c)$.

6. Find c given that the graph of $f(x)=cx^2+1/x^2$ has a point of inflection at $(1,f(1))$.

7. Find a and b given that the graph of $f(x)=ax^3+bx^2$ passes through $(-1,1)$ and has a point of inflection at $x=1/3$.

8. Determine a and b so that the curve $y=a\sqrt{x}+b/\sqrt{x}$ will have a point of inflection at $(1,4)$.

9. Find a function f such that $f'(x)=3x^2-6x+3$ and $(1,-2)$ is a point of inflection of the graph of f.

10. Find the inflection points, if any.

a) $f(x)=(x-a)^3$ b) $f(x)=(x-a)^4$

11. Show that the cubic curve $y=x^3+ax^2+bx+c$ has three possible shapes, depending on whether $a^2<3b$, $a^2=3b$, or $a^2>3b$. Sketch the three possible shapes.

12. Determine the values of a and b so that the cubic function $f(x)=x^3+ax^2+x+b$ has a point of inflection at $(-1,4)$ and then determine the equation of the tangent line at the point of inflection.

13. Consider the cubic function $f(x)=ax^3+bx^2+cx+d$. Determine the values of the coefficients a, b, c, and d such that f has a point of inflection at the origin and a local maximum at the point $(2,4)$.

14. Determine the point of intersection of the tangent lines to the points of inflection of the curve $y=x^4-6x^2+3$.

Answers

A1. a) upward on $(-\infty,\infty)$, no inflection points
b) downward on $(-\infty,0)$ and upward on $(0,\infty)$, inflection point: $(0,0)$
c) upward on $(-\infty,-1)$ or $(1,\infty)$ and downward on $(-1,1)$, inflection points: $(-1,-5)$ and $(1,-5)$
d) upward on $(-\infty,1/2)$ or $(1,\infty)$ and downward on $(1/2,1)$, inflection points: $(1/2,-1/16)$ and $(1,0)$
e) upward on $(-\infty,0)$ or $(2,\infty)$ and downward on $(0,2)$, inflection points: $(0,2)$ and $(2,6)$
f) e) upward on $(-\infty,-1)$ or $(-1/\sqrt{5},1/\sqrt{5})$ or $(1,\infty)$ and downward on $(-1,-1/\sqrt{5})$ or $(1/\sqrt{5},1)$, inflection points: $(-1,0)$, $(-1/\sqrt{5},-64/125)$, $(1/\sqrt{5},-64/125)$ and $(1,0)$
g) downward on $(-\infty,0)$ and upwnward on $(0,\infty)$, no inflection points
h) upward on $(-\infty,-1)$ or $(0,\infty)$ and downward on $(-1,0)$, inflection point: $(-1,0)$
i) upwnward on $(-\infty,0)$ or $(0,\infty)$, no inflection points
j) downward on $(-\infty,0)$ and upwnward on $(0,\infty)$, inflection point: $(0,0)$
k) downward on $(-\infty,-1)$ or $(0,1)$ and upwnward on $(-1,0)$ or $(1,\infty)$, inflection point: $(0,0)$
l) downward on $(-\infty,2)$ and upwnward on $(2,\infty)$, no inflection points
m) upwnward on $(-\infty,0)$ and downward on $(0,\infty)$, inflection point: $(0,0)$
n) downward on $(-\infty,0)$ or $(0,\infty)$, no inflection points
o) downward on $(-\infty,-1)$ and upwnward on $(-1,\infty)$, inflection points: $(-1,0)$
p) upwnward on $(-\infty,-1/\sqrt{3})$ or $(1/\sqrt{3},\infty)$ and downward on $(-1/\sqrt{3},1/\sqrt{3})$, inflection points: $(-1/\sqrt{3},4/9)$ and $(1/\sqrt{3},4/9)$
q) upwnward on $(-\infty,-1)$ or $(0,\infty)$ and downward on $(-1,0)$, inflection points: $(-1,0)$ and $(0,-1)$
r) downward on $(-\infty,0)$ or $(0,2)$ and upwnward on $(2,\infty)$, inflection point: $(2,2/9)$
s) upwnward on $(-\infty,-1)$ or $(1,\infty)$ and downward on $(-1,1)$, no inflection points
t) upwnward on $(-\infty,-1/\sqrt{3})$ or $(1/\sqrt{3},\infty)$ and downward on $(-1/\sqrt{3},1/\sqrt{3})$, inflection points: $(-1/\sqrt{3},3/4)$ and $(1/\sqrt{3},3/4)$
u) upwnward on $(-\infty,-\sqrt{6})$ or $(0,\sqrt{6})$ and downward on $(-\sqrt{6},0)$ or $(\sqrt{6},\infty)$, inflection points: $(-\sqrt{6},5/(6\sqrt{6}))$ and $(\sqrt{6},-5/(6\sqrt{6}))$

2. a) downwnward on $(-\infty,0)$ and upwnward on $(0,1)$, no concavity on $(1,\infty)$, inflection point: $(0,0)$
b) downwnward on $(2,\infty)$, no inflection point

3. a) downwnward on $(-\infty,-2)$ and upwnward on $(-2,\infty)$, inflection point: $(-2,-2e^{-2})$
b) downwnward on $(-\infty,2)$ and upwnward on $(2,\infty)$, inflection point: $(2,2e^{-2})$
c) upwnward on $(-\infty,a)$ or (b,∞) and downward on (a,b), inflection points: (a,a^2e^a) and (b,b^2e^b) where $a=-2-\sqrt{2}$ and $b=-2+\sqrt{2}$
d) upwnward on $(0,\infty)$, no inflection points
e) downwnward on $(0,e^{-3/2})$ and upwnward on $(e^{-3/2},\infty)$, inflection point: $(e^{-3/2},(-3/2)e^{-3})$
f) downwnward on $(2n\pi-\pi/2,2n\pi+\pi/2)$ and upwnward on $(2n\pi+\pi/2,2n\pi+3\pi/2)$, inflection points: $(n\pi+\pi/2,n\pi+\pi/2)$ where n is any integer
g) a) upwnward on $(-\infty,0)$ and downwnward on $(0,1)$, inflection point: $(0,0)$
h) downwnward on $(-\infty,-1)$ or $(1,\infty)$ and upward on $(-1,1)$, inflection points: $(-1,\ln 2)$ and $(1,\ln 2)$

B1. a) Lm: $(1,-1)$
b) LM: $(0,0)$, Lm: $(4,-32)$
c) LM: $(-1,4)$, Lm: $(1,0)$
d) LM: $(0,-5)$, Lm: $(-\sqrt{3},-14)$ and $(\sqrt{3},-14)$
e) Lm: $(6,-432)$
f) LM: $(-1,-2)$, Lm: $(1,2)$

g) LM: $(1,1/2)$, Lm: $(-1,-1/2)$
h) Lm: $(-1,-1/4)$
i) LM: $(0,-1)$, Lm: $(-1,-2)$ and $(1,-2)$
j) LM: $(-8,4)$, Lm: $(0,0)$
k) Lm: $(-2,-15)$ and $(3,16.25)$, LM: $(2,17)$

CQ1. a) concave upward on $(-1,\infty)$, no inflection points
b) concave upward on $(-\infty,0)$ or $(2,\infty)$, concave downward on $(0,2)$, inflection points: $(0,0)$ and $(2,6\sqrt[3]{2})$
c) concave downward on $(-\infty,0)$ or $(0,1)$, concave upward on $(1,\infty)$, inflection points: $(0,0)$ and $(1,6)$
d) concave upward on $(-2,0)$, concave downward on $(0,2)$, inflection points: $(0,0)$
e) concave downward on $(-\infty,0)$, concave upward on $(0,\infty)$, inflection points: $(0,0)$
f) concave downward on $(-\infty,0)$ or $(0,6)$, concave upward on $(6,\infty)$, inflection points: $(6,0)$
g) concave upward on $(-\infty,-\sqrt{3})$ or $(\sqrt{3},\infty)$, cancave downward on $(-\sqrt{3},-1)$ or $(-1,1)$ or $(1,\sqrt{3})$, inflection points: $(-\sqrt{3},\sqrt[3]{4})$ and $(\sqrt{3},\sqrt[3]{4})$
h) concave downward on $(-\infty,-1)$ or $(0,\infty)$, concave upward on $(-1,0)$, inflection points: $(-1,0)$ and $(0,0)$
i) concave downward on $(-\infty,(5-\sqrt{7})/6)$ or $(1,(5+\sqrt{7})/6)$, concave upward on $((5-\sqrt{7})/6,1)$ or $((5+\sqrt{7})/6,\infty)$, inflection points: $(a, f(a))$ and $(1,0)$ where $a=(5\pm\sqrt{7})/6$
j) concave downward on $(-1,-2/3)$, concave upward on $(-\infty,-1)$ or $(-2/3,\infty)$, inflection points: $(-1,1)$ and $(-2/3,0)$
k) concave downward on $(-\infty,-1)$ or $(1,\infty)$, concave upward on $(-1,1)$, inflection points: $(-1,0)$ and $(1,0)$
l) concave downward on $(-1,0)$ or $(1,\infty)$, concave upward on $(-\infty,-1)$ or $(0,1)$, inflection points: $(-1,-\sqrt[3]{2})$ and $(1,\sqrt[3]{2})$
m) concave downward on $(0,1)$ or $(1,\infty)$, concave upward on $(-\infty,-1)$ or $(-1,0)$, inflection points: $(0,0)$
n) concave downward on $(-\infty,-2)$ or $(0,\infty)$, concave upward on $(-2,0)$, inflection points: $(-2,-6\sqrt[3]{2})$ and $(0,0)$

2. $f(x)=\begin{cases} x^2, & x\ge 0 \\ -x^2, & x<0 \end{cases}$, $f''(x)=\begin{cases} 1, & x\ge 0 \\ -1, & x<0 \end{cases}$, $f''(1)$ DNE
3. a) $c\in(-\infty,-2\sqrt{2/3})\cup(2\sqrt{2/3},\infty)$
b) $c\in\{-2\sqrt{2/3},2\sqrt{2/3}\}$
c) $c\in(-2\sqrt{2/3},2\sqrt{2/3})$
4. a) neither b) yes
5. $d=(a+b+c)/3$
6. $c=-3$
7. $a=-1/3$ and $b=2/3$
8. $a=3$ and $b=1$
9. $f(x)=x^3-3x^2+3x-3$
10. a) inflection point: $(a,0)$ b) no inflection point
11. $f'(x)=3x^2+2ax+b$, $\Delta=4(a^2-3b)$
12. $a=b=3$, $y=-2x+2$
13. $a=-1/4$, $b=0$, $c=3$, $d=0$
14. $(0,6)$

