2.5 Chain Rule

A. Chain Rule

(*Chain Rule*) If g is differentiable at x and f is differentiable at $g(x)$, then the composition $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x))g'(x) \qquad (1)$$

If $u = g(x)$ and $y = f(u)$ then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad (2)$$

or

$$\frac{d}{dx} f(g(x)) = \frac{d}{du} f(u) \frac{d}{dx} g(x)$$

or

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx} \qquad (3)$$

or in words:
"The derivative of $f(u)$ is the derivative of the outside function evaluated at the inside function times the derivative of the inside function".

B. Generalized Differentiation Rules

(*Argument is a Function*) By replacing the argument x by $f(x)$ and by using the chain rule, we get generalized differentiation rules:

$$\begin{aligned}
&[(f(x))^n]' = n(f(x))^{n-1} f'(x) \\
&[\sin f(x)]' = \cos(f(x)) f'(x) \\
&[\cos f(x)]' = -\sin f(x) f'(x) \\
&[e^{f(x)}]' = e^{f(x)} f'(x) \qquad (4) \\
&[\ln f(x)]' = \frac{f'(x)}{f(x)} \\
&[a^{f(x)}]' = a^{f(x)} (\ln a) f'(x) \\
&[\log_a f(x)]' = \frac{f'(x)}{(\ln a) f(x)}
\end{aligned}$$

Practice Questions

A. Chain Rule

1. For each case, use $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ to find the derivative of $y = f(g(x))$.

a) $y = u^5$, $u = 2x - 1$

b) $y = \sqrt{u}$, $u = x^2 + 1$

c) $y = u + \frac{1}{\sqrt{u}}$, $u = x + x^2$

d) $y = u + u^2$, $u = x + \frac{1}{x}$

e) $y = u + \sqrt{u}$, $u = x + \sqrt{x}$

f) $y = \sqrt[3]{u}$, $u = 1 + x + x^2$

2. For each case, use $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ to find the derivative of $y = f(g(x))$.

a) $y = \sin u$, $u = x^2$

b) $y = \cos u$, $u = 2x + 3$

c) $y = \ln u$, $u = x^2 + 1$

d) $y = e^u$, $u = x^2$

e) $y = \log u$, $u = \sqrt{x}$

f) $y = 2^u$, $u = 2x^2$

g) $y = \tan u$, $u = \frac{x+1}{x-1}$

3. Use $\left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{du}\right|_{u=g(x)} \left.\frac{du}{dx}\right|_{x=a}$

to find $\frac{dy}{dx}$ at the given value of x.

a) $y = 2u^2 - u^3$, $u = x - \sqrt{x}$, $x = 4$

b) $y = \sqrt{1 + u^2}$, $u = \frac{x+1}{2x+1}$, $x = 1$

c) $y = u + \frac{50}{u}$, $u = 3x - \sqrt{x}$, $x = 4$

4. Use $(f \circ g)'(a) = f'(g(a))g'(a)$ to find $(f \circ g)'$ at the given point.

a) $x = 0$, $g(0) = 1$, $g'(0) = -2$, $f'(1) = 3$

a) $x = 1$, $g(1) = 0$, $g'(1) = 3$, $f'(0) = -4$

B. Generalized Differentiation Rules

1. Use the generalized differentiation rules to find the derivative of each function.

a) $y = (1 + x + x^2)^{10}$

b) $y = (x + \sqrt{x})^7$

c) $y = \sqrt{x^2 - 2x}$

d) $y = \sqrt{x + \sqrt{x}}$

e) $y = \frac{1}{\sqrt[3]{1 - x^4}}$

2. Use the generalized differentiation rules to find the derivative of each function.

a) $y = \sin\sqrt{x}$

b) $y = \sqrt{\cos x}$

c) $y = \ln\sqrt{x}$

d) $y = \sqrt{x + e^x}$

e) $y = \sin(\cos x)$

f) $y = \sqrt{\ln x}$

g) $y = \cos(\sqrt[3]{x})$

h) $y = \log(\sin x)$

i) $y = 3^{\sin x}$

j) $y = \log_2 \sqrt{x}$

k) $y = \sin\left(\frac{1}{x}\right)$

l) $y = \cos\left(\frac{1}{x^2}\right)$

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3. Find the equation of the tangent line to each graph at the given point.

a) $y=\sqrt{x^2+16}$ at $(3,5)$

b) $y=\dfrac{x}{\sqrt{1+x^2}}$ at $(0,0)$

c) $y=\dfrac{x^2}{x^2+1}$ at $\left(1,\dfrac{1}{2}\right)$

d) $y=\dfrac{\sqrt{x-1}}{x^2+1}$ at $\left(2,\dfrac{1}{5}\right)$

e) $y=\dfrac{|x|}{\sqrt{2-x^2}}$ at $(1,1)$

4. If $f(x)=1+x$, $g(x)=\dfrac{1}{1+x}$, $h(x)=\sqrt{1+x}$, determine the derivative of each composition.

a) $f\circ g\circ h$ b) $g\circ h\circ f$

c) $g\circ g$ c) $h\circ h$

5. For each case, find the intervals where the function is increasing or decreasing.

a) $f(x)=(1-x^2)^2$ b) $f(x)=x(1-x^2)^3$

c) $f(x)=\dfrac{1}{(1+x^2)^2}$ d) $f(x)=\dfrac{x}{1+x^2}$

6. Find all points on the graph of the function $y=2\sin x+\sin^2 x$ at which the tangent line is horizontal.

7. Find all points on the graph of the function $y=\sin 2x-2\sin x$ at which the tangent line is horizontal.

Challenge Questions

1. Differentiate.

a) $y=(1-2x)^3(x-3x^2)^5$ b) $y=\dfrac{(1-x+2x^2)^2}{(1-2x^2)^3}$

c) $y=[(x^2-1)^3+x^2]^2$ d) $y=\sqrt{x+\sqrt{x+\sqrt{x}}}$

e) $y=\sqrt{x+\dfrac{1}{(x-1)^2}}$ f) $y=\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$

g) $y=\dfrac{\sqrt{1-x}}{\sqrt[3]{1-x^2}}$

2. Differentiate.

a) $y=x^3\sin\left(\dfrac{1}{x}\right)$ a) $y=\sin(\cos(x^2))$

b) $y=\ln(\sin(e^x))$ c) $y=\sin(\ln\sqrt{e^x})$

d) $y=\ln(\sin(e^{\frac{1}{x}}))$ e) $y=x\tan\left(\dfrac{1}{x}\right)$

3. Find the values of x where the tangent line to the curve $y=x\sqrt{1-x^2}$ is either horizontal or vertical.

4. Analyse the continuity and differentiability at $x=0$ for:

a) $f(x)=\begin{cases} x\sin\left(\dfrac{1}{x}\right) & if\ x\neq 0\\ 0\ if\ x=0\end{cases}$

b) $f(x)=\begin{cases} x^2\sin\left(\dfrac{1}{x}\right) & if\ x\neq 0\\ 0\ if\ x=0\end{cases}$

5. Let f be a differentiable function. Use the chain rule to show that:

a) if f is even, then f' is odd

b) if f is odd, then f' is even

6. Use the chain rule to show that if f and g are inverse functions, then:

$$f'(g(x))=\frac{1}{g'(x)},\quad g'(x)\neq 0$$

7. Use the previous result to find the derivative of $\arcsin(x)$ and $\arccos(x)$.

8. Find a formula similar to (1) for the derivative of the composition $f\circ g\circ h$.

9. Given $h(1)=0$, $h'(1)=-1$, $g(0)=2$, $g'(0)=3$, $f(2)=-3$, $f'(2)=4$ find $(f\circ g\circ h)'(1)$.

10. If $y=f(u)$, $u=g(v)$, and $v=h(x)$, find a formula for $\dfrac{dy}{dx}$ similar to (2).

11. If $y=4u^2-2u$, $u=\dfrac{v+1}{v^2+1}$, and $v=\sqrt{x^2-3}$, find the value of $\left.\dfrac{dy}{dx}\right|_{x=2}$.

Answers

A1. a) $y'=10(2x-1)^4$ b) $y'=\dfrac{x}{\sqrt{x^2+1}}$

c) $y'=\left(1-\dfrac{1}{2\sqrt{x+x^2}^3}\right)(1+2x)$

d) $y'=\left(1+2\left(x+\dfrac{1}{x}\right)\right)\left(1-\dfrac{1}{x^2}\right)$

e) $y'=\left(1+\dfrac{1}{2\sqrt{x+\sqrt{x}}}\right)\left(1+\dfrac{1}{2\sqrt{x}}\right)$

f) $y'=\dfrac{1+2x}{3\sqrt[3]{(1+x+x^2)^2}}$

2. a) $y'=(2x)\cos(x^2)$ b) $y'=-2\sin(2x+3)$

c) $y'=\dfrac{2x}{x^2+1}$ d) $y'=2xe^{x^2}$

e) $y'=\dfrac{1}{2\ln 10\sqrt{x}\log\sqrt{x}}$ f) $y'=(4x\ln 2)2^{2x^2}$

g) $y'=\dfrac{-2}{(x-1)^2\cos^2\left(\dfrac{x+1}{x-1}\right)}$

3. a) -1 b) $(-2/9)/\sqrt{13}$ c) $11/8$

4. a) $(f\circ g)'(0)=-6$ b) $(f\circ g)'(1)=-12$

B1. a) $y=10(1+x+x^2)^9(1+2x)$

b) $y=7(x+\sqrt{x})^6(1+1/(2\sqrt{x}))$ c) $y=\dfrac{x-1}{\sqrt{x^2-2x}}$

d) $y=\dfrac{1+1/(2\sqrt{x})}{2\sqrt{x+\sqrt{x}}}$ e) $y=\dfrac{4x^3}{3(1-x^4)^{4/3}}$

2. a) $y=\dfrac{\cos\sqrt{x}}{2\sqrt{x}}$ b) $y=\dfrac{-\sin x}{2\sqrt{\cos x}}$

c) $y=\dfrac{1}{2x}$ d) $y=\dfrac{1+e^x}{2\sqrt{x+e^x}}$

e) $y=(-\sin x)\cos(\cos x)$ f) $y=\dfrac{1}{2x\sqrt{\ln x}}$

g) $y=\dfrac{-\sin(\sqrt[3]{x})}{3\sqrt[3]{x^2}}$ h) $y=\dfrac{\cos x}{(\ln 10)\sin x}$

i) $y=(\ln 3)(\cos x)3^{\sin x}$ j) $y=\dfrac{1}{(2\ln 2)x}$

k) $y=\left(\dfrac{-1}{x^2}\right)\cos\left(\dfrac{1}{x}\right)$ l) $y=\left(\dfrac{2}{x^3}\right)\sin\left(\dfrac{1}{x^2}\right)$

3. a) $y=(3x-16)/5$ b) $y=x$ c) $y=x/2$

d) $y=(-3x+16)/50$ e) $y=2x+1$

4. a) $(f\circ g\circ h)'(x)=\dfrac{-1}{2\sqrt{1+x}(1+\sqrt{1+x})^2}$

b) $(g\circ h\circ f)'(x)=\dfrac{-1}{(3+x)^2}$

c) $(g\circ g)'(x)=\dfrac{1}{(2+x)^2}$

d) $(h\circ h)'(x)=\dfrac{1}{4\sqrt{1+x}\sqrt{1+\sqrt{1+x}}}$

5. a) $f(x)=-4x(1-x^2)$, f is increasing on $(-1,0)\cup(1,\infty)$ and decreasing on $(-\infty,-1)\cup(0,1)$

b) $f'(x)=(1-x^2)^2(1-7x^2)$, f is increasing on $(-1/\sqrt{7},1/\sqrt{7})$ and decreasing on $(-\infty,-1)\cup(-1,-1/\sqrt{7})\cup(1/\sqrt{7},1)\cup(1,\infty)$

c) $f'(x)=\dfrac{-4x}{(1+x^2)^3}$, f is increasing if $x<0$ and decreasing if $x>0$

d) $f'(x)=\dfrac{1-x^2}{(1+x^2)^2}$, f is increasing on $(-1,1)$ and decreasing on $(-\infty,-1)\cup(1,\infty)$

6. $(\pi/2+2n\pi,3)$ and $(3\pi/2+2n\pi,-1)$ where n is an integer

7. $(\pi/3+2n\pi,-\sqrt{3}/2)$ and $(5\pi/3+2n\pi,\sqrt{3}/2)$ where n is an integer

CQ1. a) $y=x^4(1-2x)^2(1-3x)^4(78x^2-46x+5)$

b) $y'=\dfrac{2(6x-1)(1-x+2x^2)}{(1-2x^2)^4}$

c) $y'=4x[(x^2-1)^3+x^2][3(x^2-1)^2+1]$

d) $y'=\dfrac{1+\dfrac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}$ e) $y'=\dfrac{1+\dfrac{-2}{(x-1)^3}}{\sqrt{x+\dfrac{1}{(x-1)^2}}}$

f) $y'=\dfrac{-4x}{\sqrt{x^2+1}\sqrt{x^2-1}(\sqrt{x^2+1}+\sqrt{x^2-1})^2}$

g) $y'=\dfrac{-x^2+4x-3}{6\sqrt{1-x}(1-x^2)^{4/3}}$

2. a) $y'=3x^2\sin\left(\dfrac{1}{x}\right)-x\cos\left(\dfrac{1}{x}\right)$

b) $y'=-2x\sin(x^2)\cos(\cos(x^2))$

c) $y'=\dfrac{e^x\cos(e^x)}{\sin(e^x)}$ d) $y=\dfrac{1}{2}\cos\left(\dfrac{x}{2}\right)$

e) $y=\dfrac{-e^{\frac{1}{x}}}{x^2}\tan\left(e^{\frac{1}{x}}\right)$ f) $y'=\tan\left(\dfrac{1}{x}\right)-\dfrac{1}{x\cos^2\left(\dfrac{1}{x}\right)}$

3. $y'=\dfrac{1-2x^2}{\sqrt{1-x^2}}$, horizontal tangents at $\pm 1/\sqrt{2}$, vertical, vertical tangents at ± 1

4. a) $f'(x)=\sin\left(\dfrac{1}{x}\right)-\dfrac{1}{x}\cos\left(\dfrac{1}{x}\right)$, $x\neq 0$, continuous and not differentiable at $x=0$

b) $f'(x)=2x\sin\left(\dfrac{1}{x}\right)-\cos\left(\dfrac{1}{x}\right)$, $x\neq 0$, continuous and not differentiable at $x=0$

7. $\arcsin'(x)=\dfrac{1}{\sqrt{1-x^2}}$, $\arccos'(x)=\dfrac{-1}{\sqrt{1-x^2}}$

8. $(f\circ g\circ h)'(x)=f'(g(h(x)))g'(h(x))h'(x)$ **9.** $(f\circ g\circ h)'(1)=f'(g(h(1))g'(h(1)h'(1)=(4)(3)(-1)=-12$

10. $\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dv}\dfrac{dv}{dx}$

11. -6