A. Power Rule

(Power Function) The power function is defined by:
\[ f(x) = x^\alpha, \quad \alpha \in \mathbb{R} \]
The domain and the range of the power function depend on the values or the exponent \( \alpha \).

(Power Rule) The derivative of the power function \( f(x) = x^\alpha \) is given by:
\[ (x^\alpha)' = \alpha x^{\alpha-1}, \quad x \neq 0 \text{ if } \alpha < 1 \]

(Natural Exponent) If \( \alpha \) is a natural number then the power function and its derivative are defined on \( \mathbb{R} \).

(Integer Exponent) If \( \alpha \) is an integer number then the power function and its derivative are defined on \( \mathbb{R} \setminus \{0\} \) if \( \alpha < 0 \).

(Radicals) The following relation allows the convention between the radical notation and the exponential notation:
\[ n\sqrt[m]{x} = x^{\frac{m}{n}}, \quad n \text{ is natural} \]

(Rational Exponents) If the exponent \( \alpha \) is a rational number then the domain of the power function and its derivative depend on the values of \( n \) and \( m \).

(Irrational Exponent) If the exponent \( \alpha \) is an irrational number then the domain of the power function and its derivative is \( (0, \infty) \).

B. Other Basic Differentiation Rules

(Exponential Rule) The derivative of the exponential function \( f(x) = e^x \) is given by:
\[ (e^x)' = e^x \]
The domain of the exponential function and its derivative is \( (-\infty, \infty) \).

(Logarithmic Rule) The derivative of the logarithmic function \( f(x) = \ln x \) is given by:
\[ (\ln x)' = \frac{1}{x} \]
The domain of the exponential function and its derivative is \( (0, \infty) \).

(Trigonometric Rules) The derivatives of the sine and cosine functions are given by:
\[ (\sin x)' = \cos x \]
\[ (\cos x)' = -\sin x \]
The domain of the sine and cosine functions and their derivatives is \( (-\infty, \infty) \).

C. More Basic Differentiation Rules

(Constant Rule) The derivative of the constant function \( f(x) = c \) is equal to zero:
\[ c'=0 \]

(Constant Multiple Rule) If \( f \) is differentiable at \( x \) and \( c \) is any real number, then \( cf \) is also differentiable at \( x \) and:
\[ (cf(x))' = cf'(x) \]

(Sum and Difference Rule) If \( f \) and \( g \) are differentiable at \( x \), then so are \( f + g \) and \( f - g \) and:
\[ (f(x) + g(x))' = f'(x) + g'(x) \]
\[ (f(x) - g(x))' = f'(x) - g'(x) \]

Practice Questions

A. Power Rule
1. Use the power rule to differentiate.
   a) \( f(x) = x \)
   b) \( f(x) = x^2 \)
   c) \( f(x) = x^3 \)
   d) \( f(x) = x^5 \)
   e) \( f(x) = x^{10} \)
   f) \( f(x) = x^{25} \)
2. Use the power rule to differentiate.
   a) \( f(x) = x^0 \)
   b) \( f(x) = x^{-1} \)
   c) \( f(x) = \frac{1}{x^2} \)
   d) \( f(x) = x^{-3} \)
   e) \( f(x) = \frac{1}{x^5} \)
   f) \( f(x) = x^{-10} \)
3. Convert the radical notation to the exponential notation (if necessary), then use the power rule to differentiate. Specify the domain for the power function and its derivative.
   a) \( f(x) = \sqrt{x} \)
   b) \( f(x) = \sqrt[3]{x} \)
   c) \( f(x) = \sqrt[4]{x} \)
   d) \( f(x) = \sqrt[5]{x} \)
   e) \( f(x) = \sqrt[6]{x} \)
   f) \( f(x) = \sqrt[7]{x} \)
   g) \( f(x) = \frac{1}{\sqrt{x}} \)
   h) \( f(x) = \frac{1}{\sqrt[3]{x}} \)
   i) \( f(x) = \frac{1}{\sqrt[5]{x}} \)
4. Use the exponent rules to simplify, then use the power rule to differentiate.
   a) \( f(x) = x\sqrt{x} \)
   b) \( f(x) = \frac{\sqrt{x}}{x^2} \)
   c) \( f(x) = x\sqrt[3]{x} \)
   d) \( f(x) = \sqrt{x}\sqrt[3]{x} \)
   e) \( f(x) = (x^{1/2})(x^{2/3}) \)
   f) \( f(x) = \frac{\sqrt{x}}{x^2} \)
5. Use the power rule to differentiate.
   a) \( f(x) = x^{\sqrt{2}} \)
   b) \( f(x) = x^e \)
   c) \( f(x) = x^\pi \)
   d) \( f(x) = x^{1/\sqrt{3}} \)
6. Determine whether \( f \) has a vertical tangent or a cusp at \((0,0)\).
   a) \( f(x) = x^{1/3} \)  
   b) \( f(x) = x^{5/3} \)  
   c) \( f(x) = x^{2/5} \)  
   d) \( f(x) = x^{1/4} \)  
   e) \( f(x) = x^{3/2} \)  
   f) \( f(x) = x^{7/4} \)

B. Other Basic Differentiation Rules

1. Find a derivative rule for the derivative of the function \( f(x) = a^x \).
   Hint: Use the identity: \( a^x = e^{x \ln a} \)

2. Find a rule for the derivative of the function \( f(x) = \log_a x \).
   Hint: Use the identity: \( \log_a x = \frac{\ln x}{\ln a} \)

C. More Basic Differentiation Rules

1. Differentiate.
   a) \( f(x) = -2 \)  
   b) \( f(x) = 3x \)  
   c) \( f(x) = -4x^2 \)  
   d) \( f(x) = 5\sqrt{x} \)  
   e) \( f(x) = -\frac{5}{\sqrt{x}} \)  
   f) \( f(x) = 2x^2 \sqrt{x^{3/2}} \)

2. Differentiate.
   a) \( f(x) = -3\sin x \)  
   b) \( f(x) = 5\cos x \)  
   c) \( f(x) = -4e^x \)  
   d) \( f(x) = -2\ln x \)  
   e) \( f(x) = -(2)3^x \)  
   f) \( f(x) = 3\log x \)

3. Differentiate.
   a) \( f(x) = 1 - 2x + 3x^2 \)  
   b) \( f(x) = x + \frac{2}{x} - \frac{3}{x^2} + x^3 \)  
   c) \( f(x) = 2\sqrt{x} - 6\sqrt{x} \)  
   d) \( f(x) = \frac{-2}{\sqrt{x}} + \frac{3}{\sqrt[3]{x}} \)  
   e) \( f(x) = -2 + 3x^{1/2} - 6x^{-2/3} \)

4. Differentiate.
   a) \( y = (x-1)(2x+3) \)  
   b) \( y = \left( x - \frac{1}{x} \right) \left( x + \frac{2}{x} \right) \)  
   c) \( y = (\sqrt{x} - 2)(\sqrt{x} + 2) \)  
   d) \( y = (\sqrt[3]{x} + 1)(\sqrt[3]{x} - 2) \)  
   e) \( y = (2x - 1)^2 \)  
   f) \( y = \frac{2x^2 - 3x + 1}{x} \)

5. Differentiate.
   a) \( f(x) = \sin x - \cos x \)  
   b) \( f(x) = -2\cos x - 3\sin x \)  
   c) \( f(x) = -2e^x + 3\ln x \)  
   d) \( f(x) = -(2)3^x - 3\log_3 x \)

D. Tangent and Normal Lines

1. Find the equation of the tangent line to the curve \( y = x^3 - 3x^2 \) at the point \( T(1,-2) \).

2. Find the equation of the tangent line(s) with the slope \( m = -6 \) to the curve \( y = x^4 - 2x \).

3. Find the equation of the tangent line to the curve \( y = x + 1/x \) and that passes through the point \( P(2,-2) \).

4. At what points on the parabola \( xy = 12 \) is the tangent line parallel to the line \( 3x + y = 0 \).

Challenge Questions

1. For each case, find a function \( y = f(x) \) satisfying the given conditions.
   a) \( f(0) = 0, \ f'(x) = x^2 \)  
   b) \( f(0) = 1, \ f'(x) = 2\sqrt{x} \)  
   c) \( f(1) = 0, \ f'(x) = \frac{1}{x^2} \)  
   d) \( f(0) = 1, \ f'(x) = \sin x + \cos x \)  
   e) \( f(0) = 1, \ f'(x) = -2e^x \)  
   f) \( f(0) = 2, \ f'(x) = x^3 - 2x^2 + 3x - 4 \)

2. Differentiate the polynomial function:
   \( P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 \)

3. Find \( a \) and \( b \) given that the derivative of
   \( f(x) = \begin{cases} 
   ax^3 + bx^2 + 2, & x \leq 2 \\
   bx^2 - a, & x > 2 
   \end{cases} \)
   is continuous for all real \( x \).

4. Determine the coefficients \( a,b,c \) so that the curve \( f(x) = ax^2 + bx + c \) will pass through the point \( P(1,3) \) and be tangent to the line \( 4x + y = 8 \) at the point \( T(2,0) \).

5. Let \( f(x) = x^3 \). Determine the equation of the tangent line at the point \( P(c,c^3), c \neq 0 \) and another intersection (if exists) between this tangent and graph of the function \( f(x) = x^3 \).

6. Find the area of the triangle formed by the x-axis and the lines tangent and normal to the curve \( f(x) = 6x - x^2 \) at the point \( P(5,5) \).

7. Find conditions on \( a,b,c \) and \( d \) that will guarantee that the graph of the cubic polynomial
   \( p(x) = ax^3 + bx^2 + cx + d \) has:
   a) exactly two horizontal tangents  
   b) exactly one horizontal tangent  
   c) no horizontal tangent

8. Let \( f(x) = |x^2 - 4| \).
   a) study the continuity of the function \( y = f(x) \)  
   b) find a formula for the derivative function \( f'(x) \)  
   c) study the differentiability of the function \( f \)
d) sketch on the same grid the graphs of \( f \) and its
derivative \( f' \)

9. Find the equation of the normal line to the curve
\( y = \sqrt{x} \) and that passes through the point \( P(3,6) \).

10. Where does the normal line to the parabola
\( y = x - x^2 \) at the point \( P(1,0) \) intersect the parabola a
second time?

11. Find constants \( a, b, c \) and \( d \) such that the graph of
\( f(x) = ax^3 + bx^2 + cx + d \) has horizontal tangent lines
at the points \((0,1)\) and \((1,0)\).

12. Suppose that the tangent line at a point \( P \) on the
curve \( y = x^3 \) intersects the curve again at a point \( Q \).
Show that the slope of the tangent at \( Q \) is four times
the slope of the tangent at \( P \).

Answers

A1. a) \( f'(x) = 1 \)  b) \( f'(x) = 2x \)  c) \( f'(x) = 3x^2 \)
d) \( f'(x) = 5x^4 \)  e) \( f'(x) = 10x_9 \)  f) \( f'(x) = 25x^{24} \)

2. a) \( f'(x) = 0 \)  b) \( f'(x) = -1/x^2 \)  c) \( f'(x) = -2/x^3 \)
d) \( f'(x) = -3/x^4 \)  e) \( f'(x) = -6/x^7 \)  f) \( f'(x) = -10/x^{11} \)

3. a) \( f'(x) = 1/(2\sqrt{x}) \), \( D_f = [0, \infty) \), \( D_{f'} = (0, \infty) \)
b) \( f'(x) = (1/2)x^{-2/3} \), \( D_f = \{ x \in \mathbb{R} \} \), \( D_{f'} = \{ x \neq 0 \} \)
c) \( f'(x) = (3/2)\sqrt{x} \), \( D_f = [0, \infty) \), \( D_{f'} = (0, \infty) \)
d) \( f'(x) = (2/3)x^{-1/3} \), \( D_f = \{ x \in \mathbb{R} \} \), \( D_{f'} = \{ x \neq 0 \} \)
e) \( f'(x) = (1/4)x^{-3/4} \), \( D_f = [0, \infty) \), \( D_{f'} = (0, \infty) \)
f) \( f'(x) = (2/5)x^{-3/5} \), \( D_f = \{ x \in \mathbb{R} \} \), \( D_{f'} = \{ x \neq 0 \} \)
g) \( f'(x) = (-1/2)x^{-3/2} \), \( D_f = D_{f'} = (0, \infty) \)
h) \( f'(x) = (-1/3)x^{-4/3} \), \( D_f = D_{f'} = (-\infty,0) \cup (0, \infty) \)
i) \( f'(x) = (-2/3)x^{-5/3} \), \( D_f = D_{f'} = (-\infty,0) \cup (0, \infty) \)

4. a) \( f'(x) = (3/2)\sqrt{x} \)  b) \( f'(x) = (-3/2)x^{-5/2} \)
c) \( f'(x) = (4/3)\sqrt[3]{x} \)  d) \( f'(x) = (5/6)x^{-1/6} \)
e) \( f'(x) = (7/6)\sqrt[6]{x} \)  f) \( f'(x) = (5/6)/\sqrt[6]{x} \)

5. a) \( f'(x) = \sqrt[3]{x}\sqrt{2-x} \)  b) \( f'(x) = ex^{e-1} \)
c) \( f'(x) = ax^{-1} \)  d) \( f'(x) = \frac{1}{\sqrt[3]{x}} \)

6. a) vertical tangent b) none c) vertical tangent and
cusp d) vertical tangent e) none f) vertical tangent

B1. \((a^3)^y = (ln a)a^x \)  2. \((log_a x)^y = \frac{1}{(ln a)x} \)

C1. a) \( f'(x) = 0 \)  b) \( f'(x) = 3 \)

c) \( f'(x) = -8x \)  d) \( f'(x) = (5/2)/\sqrt{x} \)
e) \( f'(x) = x^{-6/5} \)  f) \( f'(x) = (17/3)x^{11/6} \)

2. a) \( f'(x) = -3\cos x \)  b) \( f'(x) = -5\sin x \)
c) \( f'(x) = -4e^x \)  d) \( f'(x) = -2/x \)
e) \( f'(x) = (-2\ln 3)/3 \)  f) \( f'(x) = 3/(x\ln 10) \)

3. a) \( f'(x) = -2+6x \)  b) \( f'(x) = 1-2/x^2 + 6/x^3 + 3x^2 \)
c) \( f'(x) = 1/\sqrt{x} - 2x^{-2/3} \)  d) \( f'(x) = x^{-3/2} - x^{-2/3} \)
e) \( f'(x) = (3/2)/\sqrt{x} + 4x^{-5/3} \)

4. a) \( y' = 4x + 1 \)  b) \( y' = 2x + 4/x^3 \)  c) \( y' = 1 \)
d) \( y' = (5/6)/\sqrt{x} - (2/3)x^{-2/3} + (1/2)/\sqrt{x} \)
e) \( y' = 8x - 4 \)  f) \( y' = 2 - 1/x^2 \)

5. a) \( f'(x) = \cos x + \sin x \)  b) \( f'(x) = 2\sin x - 3\cos x \)
c) \( f'(x) = (-2)e^x + 3/x \)
d) \( f'(x) = (-2\ln 3)/3 - 3/(x\ln 3) \)

D1. \( y = -3x + 1 \)  2. \( y = -6x - 3 \)  3. \( y = -3x + 4 \)  4. \( y = -2 \)

4. \( A(-2,-6) \) and \( B(2,6) \)

CQ1. a) \( f(x) = x^3/3 \)  b) \( f(x) = (4/3)x\sqrt{x} \)
c) \( f(x) = 1-1/x \)  d) \( f(x) = 2 - \cos x + \sin x \)
e) \( f(x) = 3 - 2x^5 \)  f) \( f(x) = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{3}{2}x^2 - 4x + 2 \)

2. \( P(x) = \sum_{n=0}^{\infty} a_n x^{n+1} \)  \((a_n-1)x^{n-2} + \ldots + 2a_2x + a_1 \)

a) \( a = 2 \)  b) \( b = 0 \)  c) \( c = 4 \)

3. \( y = 3c^2x - 2c^3 \)  \( P(-2c,-8c^3) \)

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7. a) \( b^2 > 3ac \)  b) \( b^2 = 3ac \)  c) \( b^2 < 3ac \)

8. a) \( f \) is continuous everywhere b) \( f \) is not
differentiable at \( x = -2 \) and \( x = 2 \)

\[\begin{align*}
  &2x, \quad x < -2 \\
  &-2x, \quad -2 < x < 2 \\
  &2x, \quad x > 2
\end{align*}\]

c) \( f'(x) = \frac{1}{\sqrt{x}} \)

9. \( y = -4x + 18 \)

10. \( Q(-1,-2) \)
11. $a = 2$, $b = -3$, $c = 0$ and $d = 1$