2.2 Basic Differentiation Rules

A. Power Rule

(*Power Function)* The power function is defined by:

$$f(x) = x^{\alpha}, \quad \alpha \in \mathbb{R}$$

The domain and the range of the power function depend on the values or the exponent α .

(*Power Rule*) The derivative of the power function $f(x) = x^{\alpha}$ is given by:

$$(x^{\alpha})' = \alpha x^{\alpha - 1}, \quad x \neq 0 \; if \; \alpha < 1$$

(*Natural Exponent*) If α is a natural number then the power function and its derivative are defined on $\mathbb{R}$.

(*Integer Exponent*) If α is an integer number then the power function and its derivative are defined on $\mathbb{R} \setminus \{0\}$ if $\alpha < 0$.

(*Radicals*) The following relation allows the convention between the radical notation and the exponential notation:

$$\sqrt[n]{x^m} = x^{\frac{m}{n}}, \quad n \; is \; natural$$

(*Rational Exponents*) If the exponent α is a rational number then the domain of the power function and its derivative depend on the values of n and m .

(*Irrational Exponent*) If the exponent α is an irrational number then the domain of the power function and its derivative is $(0, \infty)$.

B. Other Basic Differentiation Rules

(*Exponential Rule*) The derivative of the exponential function $f(x) = e^x$ is given by:

$$(e^x)' = e^x$$

The domain of the exponential function and its derivative is $(-\infty, \infty)$.

(*Logarithmic Rule*) The derivative of the logarithmic function $f(x) = \ln x$ is given by:

$$(\ln x)' = \frac{1}{x}$$

The domain of the exponential function and its derivative is $(0, \infty)$.

(*Trigonometric Rules*) The derivatives of the sine and cosine functions are given by:

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

The domain of the sine and cosine functions and their derivatives is $(-\infty, \infty)$.

C. More Basic Differentiation Rules

(*Constant Rule*)
The derivative of the constant function $f(x) = c$ is equal to zero:

$$c' = 0$$

(*Constant Multiple Rule*) If f is differentiable at x and c is any real number, then cf is also differentiable at x and:

$$(cf(x))' = cf'(x)$$

(*Sum and Difference Rule*) If f and g are differentiable at x , then so are $f + g$ and $f - g$ and:

$$(f(x) + g(x))' = f'(x) + g'(x)$$

$$(f(x) - g(x))' = f'(x) - g'(x)$$

Practice Questions

A. Power Rule

1. Use the power rule to differentiate.

a) $f(x) = x$ b) $f(x) = x^2$ c) $f(x) = x^3$

d) $f(x) = x^5$ e) $f(x) = x^{10}$ f) $f(x) = x^{25}$

2. Use the power rule to differentiate.

a) $f(x) = x^0$ b) $f(x) = x^{-1}$ c) $f(x) = \frac{1}{x^2}$

d) $f(x) = x^{-3}$ e) $f(x) = \frac{1}{x^6}$ f) $f(x) = x^{-10}$

3. Convert the radical notation to the exponential notation (if necessary), then use the power rule to differentiate. Specify the domain for the power function and its derivative.

a) $f(x) = \sqrt{x}$ b) $f(x) = \sqrt[3]{x}$ c) $f(x) = \sqrt{x^3}$

d) $f(x) = \sqrt[3]{x^2}$ e) $f(x) = x^{1/4}$ f) $f(x) = x^{2/5}$

g) $f(x) = \frac{1}{\sqrt{x}}$ h) $f(x) = \frac{1}{\sqrt[3]{x}}$ i) $f(x) = x^{-2/3}$

4. Use the exponent rules to simplify, then use the power rule to differentiate.

a) $f(x) = x\sqrt{x}$ b) $f(x) = \frac{\sqrt{x}}{x^2}$

c) $f(x) = x\sqrt[3]{x}$ d) $f(x) = \sqrt{x}\sqrt[3]{x}$

e) $f(x) = (x^{1/2})(x^{2/3})$ f) $f(x) = \frac{x\sqrt{x}}{\sqrt[3]{x^2}}$

5. Use the power rule to differentiate.

a) $f(x) = x^{\sqrt{2}}$ b) $f(x) = x^e$

c) $f(x) = x^{\pi}$ d) $f(x) = x^{1/\sqrt{3}}$

6. Determine whether f has a vertical tangent or a cusp at $(0,0)$.

a) $f(x)=x^{1/3}$ b) $f(x)=x^{5/3}$ c) $f(x)=x^{2/5}$

d) $f(x)=x^{1/4}$ e) $f(x)=x^{3/2}$ f) $f(x)=x^{7/4}$

B. Other Basic Differentiation Rules

1. Find a derivative rule for the derivative of the function $f(x)=a^x$.

Hint: Use the identity: $a^x=e^{x\ln a}$

2. Find a rule for the derivative of the function $f(x)=\log_a x$.

Hint. Use the identity: $\log_a x=\dfrac{\ln x}{\ln a}$

C. More Basic Differentiation Rules

1. Differentiate.

a) $f(x)=-2$ b) $f(x)=3x$ c) $f(x)=-4x^2$

d) $f(x)=5\sqrt{x}$ e) $f(x)=\dfrac{-5}{\sqrt[5]{x}}$ f) $f(x)=2x^2\sqrt{x}\,\sqrt[3]{x}$

2. Differentiate.

a) $f(x)=-3\sin x$ b) $f(x)=5\cos x$

c) $f(x)=-4e^x$ d) $f(x)=-2\ln x$

e) $f(x)=(-2)3^x$ f) $f(x)=3\log x$

3. Differentiate.

a) $f(x)=1-2x+3x^2$ b) $f(x)=x+\dfrac{2}{x}-\dfrac{3}{x^2}+x^3$

c) $f(x)=2\sqrt{x}-6\sqrt[3]{x}$ d) $f(x)=\dfrac{-2}{\sqrt{x}}+\dfrac{3}{\sqrt[3]{x}}$

e) $f(x)=-2+3x^{1/2}-6x^{-2/3}$

4. Differentiate.

a) $y=(x-1)(2x+3)$ b) $y=\left(x-\dfrac{1}{x}\right)\left(x+\dfrac{2}{x}\right)$

c) $y=(\sqrt{x}-2)(\sqrt{x}+2)$ d) $y=(\sqrt[3]{x}+1)(\sqrt{x}-2)$

e) $y=(2x-1)^2$ f) $y=\dfrac{2x^2-3x+1}{x}$

5. Differentiate.

a) $f(x)=\sin x-\cos x$ b) $f(x)=-2\cos x-3\sin x$

c) $f(x)=-2e^x+3\ln x$ d) $f(x)=(-2)3^x-3\log_3 x$

D. Tangent and Normal Lines

1. Find the equation of the tangent line to the curve $y=x^3-3x^2$ at the point $T(1,-2)$.

2. Find the equation of the tangent line(s) with the slope $m=-6$ to the curve $y=x^4-2x$.

3. Find the equation of the tangent line to the curve $y=x+1/x$ and that passes through the point $P(2,-2)$.

4. At what points on the parabola $xy=12$ is the tangent line parallel to the line $3x+y=0$.

Challenge Questions

1. For each case, find a function $y=f(x)$ satisfying the given conditions.

a) $f(0)=0$, $f'(x)=x^2$

b) $f(0)=1$, $f'(x)=2\sqrt{x}$

c) $f(1)=0$, $f'(x)=\dfrac{1}{x^2}$

d) $f(0)=1$, $f'(x)=\sin x+\cos x$

e) $f(0)=1$, $f'(x)=-2e^x$

f) $f(0)=2$, $f'(x)=x^3-2x^2+3x-4$

2. Differentiate the polynomial function:
$P(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0$

3. Find a and b given that the derivative of

$$f(x)=\begin{cases}ax^3+bx+2, & x\le 2\\ bx^2-a, & x>2\end{cases}$$

is continuous for all real x.

4. Determine the coefficients a,b,c so that the curve $f(x)=ax^2+bx+c$ will pass through the point $P(1,3)$ and be tangent to the line $4x+y=8$ at the point $T(2,0)$.

5. Let $f(x)=x^3$. Determine the equation of the tangent line at the point $P(c,c^3), c\neq 0$ and another intersection (if exists) between this tangent and graph of the function $f(x)=x^3$.

6. Find the area of the triangle formed by the x-axis and the lines tangent and normal to the curve $f(x)=6x-x^2$ at the point $P(5,5)$.

7. Find conditions on a,b,c and d that will guarantee that the graph of the cubic polynomial $p(x)=ax^3+bx^2+cx+d$ has:

a) exactly two horizontal tangents
b) exactly one horizontal tangent
c) no horizontal tangent

8. Let $f(x)=|x^2-4|$.

a) study the continuity of the function $y=f(x)$

b) find a formula for the derivative function $f'(x)$

c) study the differentiability of the function f

d) sketch on the same grid the graphs of f and its derivative f'

9. Find the equation of the normal line to the curve $y=\sqrt{x}$ and that passes through the point $P(3,6)$.

10. Where does the normal line to the parabola $y=x-x^2$ at the point $P(1,0)$ intersect the parabola a second time?

11. Find constants a,b,c and d such that the graph of $f(x)=ax^3+bx^2+cx+d$ has horizontal tangent lines at the points $(0,1)$ and $(1,0)$.

12. Suppose that the tangent line at a point P on the curve $y=x^3$ intersects the curve again at a point Q. Show that the slope of the tangent at Q is four times the slope of the tangent at P.

Answers

A1. a) $f'(x)=1$ b) $f'(x)=2x$ c) $f'(x)=3x^2$
d) $f'(x)=5x^4$ e) $f'(x)=10x_9$ f) $f'(x)=25x^{24}$

2. a) $f'(x)=0$ b) $f'(x)=-1/x^2$ c) $f'(x)=-2/x^3$
d) $f'(x)=-3/x^4$ e) $f'(x)=-6/x^7$ f) $f'(x)=-10/x^{11}$

3. a) $f'(x)=1/(2\sqrt{x})$, $D_f=[0,\infty)$, $D_{f'}=(0,\infty)$
b) $f'(x)=(1/2)x^{-2/3}$, $D_f=\{x\in R\}$, $D_{f'}=\{x\neq 0\}$
c) $f'(x)=(3/2)\sqrt{x}$, $D_f=D_{f'}=[0,\infty)$
d) $f'(x)=(2/3)x^{-1/3}$, $D_f=\{x\in R\}$, $D_{f'}=\{x\neq 0\}$
e) $f'(x)=(1/4)x^{-3/4}$, $D_f=[0,\infty)$, $D_{f'}=(0,\infty)$
f) $f'(x)=(2/5)x^{-3/5}$, $D_f=\{x\in R\}$, $D_{f'}=\{x\neq 0\}$
g) $f'(x)=(-1/2)x^{-3/2}$, $D_f=D_{f'}=(0,\infty)$
h) $f'(x)=(-1/3)x^{-4/3}$, $D_f=D_{f'}=(-\infty,0)\cup(0,\infty)$
i) $f'(x)=(-2/3)x^{-5/3}$, $D_f=D_{f'}=(-\infty,0)\cup(0,\infty)$

4. a) $f'(x)=(3/2)\sqrt{x}$ b) $f'(x)=(-3/2)x^{-5/2}$
c) $f'(x)=(4/3)\sqrt[3]{x}$ d) $f'(x)=(5/6)x^{-1/6}$
e) $f'(x)=(7/6)\sqrt[6]{x}$ f) $f'(x)=(5/6)/\sqrt[6]{x}$

5. a) $f'(x)=\sqrt{2}x^{\sqrt{2}-1}$ b) $f'(x)=ex^{e-1}$
c) $f'(x)=\pi x^{\pi-1}$ d) $f'(x)=\dfrac{1}{\sqrt{3}}x^{\left(\frac{1}{\sqrt{3}}-1\right)}$

6. a) vertical tangent b) none c) vertical tangent and cusp d) vertical tangent e) none f) vertical tangent

B1. $(a^x)'=(\ln a)a^x$ **2.** $(\log_a x)'=\dfrac{1}{(\ln a)x}$

C1. a) $f'(x)=0$ b) $f'(x)=3$
c) $f'(x)=-8x$ d) $f'(x)=(5/2)/\sqrt{x}$
e) $f'(x)=x^{-6/5}$ f) $f'(x)=(17/3)x^{11/6}$

2. a) $f'(x)=-3\cos x$ b) $f'(x)=-5\sin x$
c) $f'(x)=-4e^x$ d) $f'(x)=-2/x$
e) $f'(x)=(-2\ln 3)3^x$ f) $f'(x)=3/(x\ln 10)$

3. a) $f'(x)=-2+6x$ b) $f'(x)=1-2/x^2+6/x^3+3x^2$
c) $f'(x)=1/\sqrt{x}-2x^{-2/3}$ d) $f'(x)=x^{-3/2}-x^{-2/3}$
e) $f'(x)=(3/2)/\sqrt{x}+4x^{-5/3}$

4. a) $y'=4x+1$ b) $y'=2x+4/x^3$ c) $y'=1$
d) $y'=(5/6)/\sqrt[6]{x}-(2/3)x^{-2/3}+(1/2)/\sqrt{x}$
e) $y'=8x-4$ f) $y'=2-1/x^2$

5. a) $f'(x)=\cos x+\sin x$ b) $f'(x)=2\sin x-3\cos x$
c) $f'(x)=(-2)e^x+3/x$
d) $f'(x)=(-2\ln 3)3^x-3/(x\ln 3)$

D1. $y=-3x+1$ **2.** $y=-6x-3$ **3.** $y=-3x+4$ or $y=-2$
4. $A(-2,-6)$ and $B(2,6)$

CQ1. a) $f(x)=x^3/3$ b) $f(x)=(4/3)x\sqrt{x}$
c) $f(x)=1-1/x$ d) $f(x)=2-\cos x+\sin x$
e) $f(x)=3-2e^x$ f) $f(x)=\dfrac{1}{4}x^4-\dfrac{2}{3}x^3+\dfrac{3}{2}x^2-4x+2$

2. $P'(x)=na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+...+2a_2x+a_1$

3. $a=-2$ and $b=-8$

4. $a=-1$, $b=0$ and $c=4$

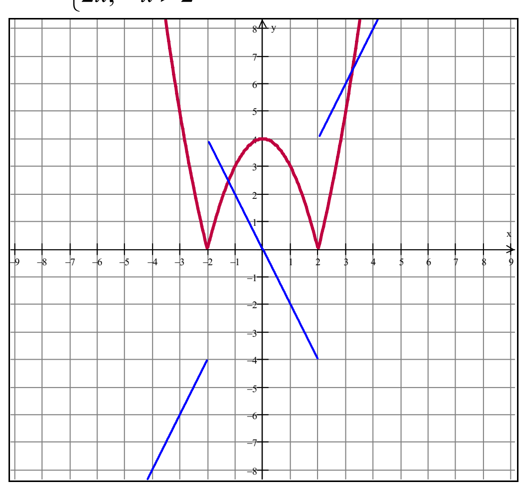
5. $y=3c^2x-2c^3$, $P(-2c,-8c^3)$

6. $425/8$

7. a) $b^2>3ac$ b) $b^2=3ac$ c) $b^2<3ac$

8. a) f is continuous everywhere b) f is not differentiable at $x=-2$ and $x=2$

c) $f'(x)=\begin{cases}2x, & x<-2\\ -2x, & -2<x<2\\ 2x, & x>2\end{cases}$ d) See the figure bellow.



9. $y=-4x+18$

10. $Q(-1,-2)$

11. $a = 2$, $b = -3$, $c = 0$ and $d = 1$