

3.4 Asymptotes

A. Vertical Asymptote

(*Infinite Limits*) If the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a with $x < a$, then $\lim_{x \rightarrow a^-} f(x) = \infty$.

If the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a with $x > a$, then $\lim_{x \rightarrow a^+} f(x) = \infty$.

If the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a , then $\lim_{x \rightarrow a} f(x) = \infty$.

If the values of $f(x)$ can be made arbitrarily large in absolute value but negative, by taking x sufficiently close to a with $x < a$, then $\lim_{x \rightarrow a^-} f(x) = -\infty$.

If the values of $f(x)$ can be made arbitrarily large in absolute value but negative, by taking x sufficiently close to a with $x > a$, then $\lim_{x \rightarrow a^+} f(x) = -\infty$.

If the values of $f(x)$ can be made arbitrarily large in absolute value but negative, by taking x sufficiently close to a , then $\lim_{x \rightarrow a} f(x) = -\infty$.

B. Vertical Asymptotes

(*Vertical Asymptote*) A vertical line $y = a$ is called a vertical asymptote of the curve $y = f(x)$ if:

$$\lim_{x \rightarrow a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^+} f(x) = \pm\infty$$

Note. A vertical asymptote may have an intersection point with the graph of the function.

(*Finding Vertical Asymptotes*) The vertical asymptotes appear at the zeros of the denominator which are not zeros of the numerator.

C. Limits at Infinity

(*Limit at Infinity*) If the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large, then $\lim_{x \rightarrow \infty} f(x) = L$.

Read: "The limit of $f(x)$ as x approaches infinity is L ".

If the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large in absolute value but negative, then $\lim_{x \rightarrow -\infty} f(x) = L$.

D. Horizontal Asymptotes

(*Horizontal Asymptote*) A horizontal line $y = L$ is called horizontal asymptote of the curve $y = f(x)$ if

$$\lim_{x \rightarrow \pm\infty} f(x) = L.$$

Note. A Horizontal asymptote may be crosses or touched by the graph of the function $y = f(x)$.

Note. A curve may have at most two horizontal asymptotes (one for $-\infty$ and one for $+\infty$).

(*Basic Limits at Infinity*) If $\alpha > 1$ then $\lim_{x \rightarrow \pm\infty} \frac{1}{x^\alpha} = 0$.

(*Rational Function*). Consider the rational function

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + \dots + a_1 x + a_0}{b_m x^m + \dots + b_1 x + b_0}.$$
 Then:

$$\lim_{x \rightarrow \pm\infty} f(x) = \begin{cases} \frac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \\ \text{sgn}\left(\frac{a_n}{b_m}\right)(\pm\infty)^{n-m} & \text{if } n > m \end{cases}$$

Note. Any rational function behaves as $(a_n / b_m)x^{n-m}$ at infinity.

E. Infinite Limits at Infinity

(*Infinite Limit at Infinity*) If the values of $f(x)$ can be made arbitrarily large by taking x sufficiently large, then $\lim_{x \rightarrow \infty} f(x) = \infty$.

Similarly, we may have:

$$\lim_{x \rightarrow -\infty} f(x) = \infty \quad \text{or} \quad \lim_{x \rightarrow \infty} f(x) = -\infty \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = -\infty$$

(*Basic Limits at Infinity*) If $\alpha > 1$ then

$$\lim_{x \rightarrow \pm\infty} x^\alpha = (\pm\infty)^\alpha.$$

F. Oblique Asymptotes

(*Oblique Asymptotes*) The line $y = mx + b$ is a linear oblique asymptote for the curve $y = f(x)$ if:

$$\lim_{x \rightarrow \pm\infty} [f(x) - (mx + b)] = 0$$

Note. An oblique asymptote may be crosses or touched by the graph of the function $y = f(x)$.

Note. A curve may have at most two oblique asymptotes (one for $-\infty$ and one for $+\infty$).

(*Rational Functions*) If $n = m + 1$ then the rational function has an oblique asymptote whose equation can be found by long division algorithm:

$$\frac{P(x)}{Q(x)} = mx + b + \frac{R(x)}{Q(x)}$$

(*Non Rational Functions*) The parameters m and b can be found by computing the following limits:

$$m = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x}$$

$$b = \lim_{x \rightarrow \pm\infty} (f(x) - mx)$$

Practice Questions

1. For each case, find the limit.

a) $\lim_{x \rightarrow 0^-} \frac{1}{x}$

b) $\lim_{x \rightarrow 0^+} \frac{1}{x}$

c) $\lim_{x \rightarrow 0} \frac{1}{x}$

d) $\lim_{x \rightarrow 0} \frac{1}{x^2}$

e) $\lim_{x \rightarrow 0^+} \frac{1}{x^2}$

f) $\lim_{x \rightarrow 0} \frac{1}{x^2}$

g) $\lim_{x \rightarrow 2^-} \frac{1}{x-2}$

h) $\lim_{x \rightarrow 2^+} \frac{1}{x-2}$

i) $\lim_{x \rightarrow 2} \frac{1}{x-2}$

j) $\lim_{x \rightarrow -1} \frac{1}{(x+1)^2}$

k) $\lim_{x \rightarrow -1^+} \frac{1}{(x+1)^2}$

l) $\lim_{x \rightarrow -1} \frac{1}{(x+1)^2}$

2. Find each limit.

a) $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$

b) $\lim_{x \rightarrow -1} \frac{x - 2}{x^2 - x - 2}$

c) $\lim_{x \rightarrow -1^+} \frac{x - 2}{x^2 - x - 2}$

d) $\lim_{x \rightarrow -1} \frac{x - 2}{x^2 - x - 2}$

e) $\lim_{x \rightarrow -2} \frac{x + 1}{(x + 2)^2}$

f) $\lim_{x \rightarrow -2} \frac{x^2 - 4}{(x + 2)^2}$

3. Find the equation of any vertical asymptote.

a) $f(x) = \frac{1}{x}$

b) $f(x) = \frac{1}{x^2}$

c) $f(x) = \frac{1}{x - 2}$

d) $f(x) = \frac{1}{(x + 1)^2}$

e) $f(x) = \frac{x}{x - 1}$

f) $f(x) = \frac{x - 1}{x^2 - 1}$

g) $f(x) = \frac{x^2 - 1}{x - 1}$

h) $f(x) = \frac{x + 1}{x^2 - x - 2}$

i) $f(x) = \frac{x^2 - 9}{(x - 3)^2}$

j) $f(x) = \frac{x^2 - 1}{x^4 - 1}$

k) $f(x) = \frac{1}{x^2 - 4}$

l) $f(x) = \frac{x + 3}{x^2 + x - 2}$

4. Sketch the graph of the function near any vertical asymptote.

a) $f(x) = \frac{1}{x - 2}$

b) $f(x) = \frac{1}{(x + 1)^3}$

c) $f(x) = \frac{1}{x^2 - 4}$

d) $f(x) = \frac{x - 2}{x^2 + 3x + 2}$

e) $f(x) = \frac{x + 1}{x^2 + 3x + 2}$

f) $f(x) = \frac{x^2 - 4}{(x + 2)^2}$

g) $f(x) = \frac{x^3 - 1}{x^2 - 1}$

h) $f(x) = \frac{x^3 - 1}{x^4 - 1}$

5. Find the limit and the equation of the horizontal asymptote, if they exist.

a) $\lim_{x \rightarrow \infty} \frac{1}{x}$

b) $\lim_{x \rightarrow -\infty} \frac{1}{x^2}$

c) $\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}}$

d) $\lim_{x \rightarrow -\infty} \frac{1}{\sqrt[3]{x}}$

e) $\lim_{x \rightarrow \infty} \frac{x}{x + 1}$

f) $\lim_{x \rightarrow \infty} \frac{x}{x^2 + 1}$

g) $\lim_{x \rightarrow \infty} \frac{1 - x^2}{x^2 + 1}$

h) $\lim_{x \rightarrow -\infty} \frac{2x^2 - 3x + 1}{-3x^2 + x - 2}$

i) $\lim_{x \rightarrow \infty} \sqrt{x}$

j) $\lim_{x \rightarrow \infty} \frac{x^2}{x + 1}$

k) $\lim_{x \rightarrow \infty} \frac{-x^3 - 3x}{-2x^3 + x^2}$

l) $\lim_{x \rightarrow -\infty} \frac{2x^4 - 1}{-x^4 + 1}$

6. Use the long division algorithm to find the equation of the oblique asymptote, if it exists.

a) $f(x) = \frac{x^2}{x + 1}$

b) $f(x) = \frac{x^3 + x}{x^2 - 1}$

c) $f(x) = \frac{x^2}{1 - x} - \frac{x^2}{1 + x}$

d) $f(x) = \frac{x^4 - 1}{x^3 - 1}$

7. Find each limit, if it exists. For each case, state the indeterminate form.

a) $\lim_{x \rightarrow \infty} (\sqrt{x + 1} - \sqrt{x})$

b) $\lim_{x \rightarrow \infty} \left(\frac{1}{x} - \frac{1}{x + 1} \right)$

c) $\lim_{x \rightarrow \infty} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x + 1}} \right)$

d) $\lim_{x \rightarrow \infty} (\sqrt{2x} - \sqrt{x})$

e) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x)$

f) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x)$

g) $\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{2\sqrt{x + 1}}$

h) $\lim_{x \rightarrow \infty} \frac{\sqrt{x + 1}}{\sqrt{4x - 1}}$

i) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x)$

j) $\lim_{x \rightarrow \pm\infty} \frac{2x}{|x| + 1}$

k) $\lim_{x \rightarrow \infty} \frac{\sqrt[3]{x}}{\sqrt{x^2 + 1}}$

8. Find the equation of any asymptote.

a) $f(x) = \frac{x + 1}{x - 1}$

b) $f(x) = \frac{x^2 + 1}{x^2 - 1}$

c) $f(x) = \frac{x^2 + 1}{x - 2}$

d) $f(x) = \frac{x}{x^2 - 4}$

e) $f(x) = \frac{x|x|}{x + 1}$

f) $f(x) = \frac{x^2 - 4}{|x| - 2}$

Answers

1. a) $-\infty$ b) ∞ c) DNE d) ∞ e) ∞ f) ∞ g) $-\infty$ h) ∞
i) DNE j) ∞ k) ∞ l) ∞

2. a) 2 b) $-\infty$ c) ∞ d) DNE e) ∞ f) DNE

3. a) $x=0$ b) $x=0$ c) $x=2$ d) $x=-1$ e) $x=1$
f) $x=-1$ g) none h) $x=2$ i) $x=3$ j) none
k) $x=\pm 2$ l) $x=1$ and $x=-2$

5. a) $0, y=0$ b) $0, y=0$ c) $0, y=0$ d) $0, y=0$
e) $1, y=1$ f) $0, y=0$ g) $-1, y=-1$ h) $-2/3, y=-2/3$
i) ∞ j) ∞ k) $1/2, y=1/2$ l) $-2, y=-2$

6. a) $y=x-1$ b) $y=x$ c) $y=-2x$ d) $y=x$

7. a) 0 b) 0 c) 0 d) ∞ e) 0 f) $1/2$ g) $1/2$ h) $1/2$
i) $1/2$ j) $\lim_{x \rightarrow \infty} f(x) = 2$ and $\lim_{x \rightarrow -\infty} f(x) = -2$ k) 0

8. a) HA: $y=1$, VA: $x=1$

b) HA: $y=1$, VA: $x=\pm 1$

c) VA: $x=2$, OA: $y=x+2$

d) VA: $x=\pm 2$, HA: $y=0$

e) VA: $x=-1$, OA: $y=x-1$ at ∞ and $y=-x+1$ at $-\infty$

f) OA: $y=x+2$ at ∞ and $y=-x+2$ at $-\infty$