3.4 Asymptotes

A. Vertical Asymptote

(*Infinite Limits*) If the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a with $x < a$, then $\lim_{x \to a^-} f(x) = \infty$.

If the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a with $x > a$, then $\lim_{x \to a^+} f(x) = \infty$.

If the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, then $\lim_{x \to a} f(x) = \infty$.

If the values of $f(x)$ can be made arbitrarily large in absolute value but negative, by taking x sufficiently close to a with $x < a$, then $\lim_{x \to a^-} f(x) = -\infty$.

If the values of $f(x)$ can be made arbitrarily large in absolute value but negative, by taking x sufficiently close to a with $x > a$, then $\lim_{x \to a^+} f(x) = -\infty$.

If the values of $f(x)$ can be made arbitrarily large in absolute value but negative, by taking x sufficiently close to a, then $\lim_{x \to a} f(x) = -\infty$.

B. Vertical Asymptotes

(*Vertical Asymptote*) A vertical line $y = a$ is called a vertical asymptote of the curve $y = f(x)$ if:

$\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$

Note. A vertical asymptote may have an intersection point with the graph of the function.

(*Finding Vertical Asymptotes*) The vertical asymptotes appear at the zeros of the denominator which are not zeros of the numerator.

C. Limits at Infinity

(*Limit at Infinity*) If the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large, then $\lim_{x \to \infty} f(x) = L$.

Read: "The limit of $f(x)$ as x approaches infinity is L".

If the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large in absolute value but negative, then $\lim_{x \to -\infty} f(x) = L$.

D. Horizontal Asymptotes

(*Horizontal Asymptote*) A horizontal line $y = L$ is called horizontal asymptote of the curve $y = f(x)$ if $\lim_{x \to \pm\infty} f(x) = L$.

Note. A Horizontal asymptote may be crosses or touched by the graph of the function $y = f(x)$.

Note. A curve may have at most two horizontal asymptotes (one for $-\infty$ and one for $+\infty$).

(*Basic Limits at Infinity*) If $\alpha > 1$ then $\lim_{x \to \pm\infty} \frac{1}{x^\alpha} = 0$.

(*Rational Function*). Consider the rational function $f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + \ldots a_1 x + a_0}{b_m x^m + \ldots b_1 x + b_0}$. Then:

$$\lim_{x \to \pm\infty} f(x) = \begin{cases} \frac{a_n}{b_m} & if\ n = m \\ 0 & if\ n < m \\ \operatorname{sgn}(\frac{a_n}{b_m})(\pm\infty)^{n-m} & if\ n > m \end{cases}$$

Note. Any rational function behaves as $(a_n / b_m)x^{n-m}$ at infinity.

E. Infinite Limits at Infinity

(*Infinite Limit at Infinity*) If the values of $f(x)$ can be made arbitrarily large by taking x sufficiently large, then $\lim_{x \to \infty} f(x) = \infty$.

Similarly, we may have:

$\lim_{x \to -\infty} f(x) = \infty$ or $\lim_{x \to \infty} f(x) = -\infty$ or $\lim_{x \to -\infty} f(x) = -\infty$

(*Basic Limits at Infinity*) If $\alpha > 1$ then $\lim_{x \to \pm\infty} x^\alpha = (\pm\infty)^\alpha$.

F. Oblique Asymptotes

(*Oblique Asymptotes*) The line $y = mx + b$ is a linear oblique asymptote for the curve $y = f(x)$ if:

$$\lim_{x \to \pm\infty} [f(x) - (mx + b)] = 0$$

Note. An oblique asymptote may be crosses or touched by the graph of the function $y = f(x)$.

Note. A curve may have at most two oblique asymptotes (one for $-\infty$ and one for $+\infty$).

(*Rational Functions*) If $n = m + 1$ then the rational function has an oblique asymptote whose equation can be found by long division algorithm:

$$\frac{P(x)}{Q(x)} = mx + b + \frac{R(x)}{Q(x)}$$

(*Non Rational Functions*) The parameters m and b can be found by computing the following limits:

$$m = \lim_{x \to \pm\infty} \frac{f(x)}{x}$$

$$b = \lim_{x \to \pm\infty} (f(x) - mx)$$

Practice Questions

1. For each case, find the limit.

a) $\lim_{x\to 0^-}\frac{1}{x}$ b) $\lim_{x\to 0^+}\frac{1}{x}$

c) $\lim_{x\to 0}\frac{1}{x}$ d) $\lim_{x\to 0^-}\frac{1}{x^2}$

e) $\lim_{x\to 0^+}\frac{1}{x^2}$ f) $\lim_{x\to 0}\frac{1}{x^2}$

g) $\lim_{x\to 2^-}\frac{1}{x-2}$ h) $\lim_{x\to 2^+}\frac{1}{x-2}$

i) $\lim_{x\to 2}\frac{1}{x-2}$ j) $\lim_{x\to -1^-}\frac{1}{(x+1)^2}$

k) $\lim_{x\to -1^+}\frac{1}{(x+1)^2}$ l) $\lim_{x\to -1}\frac{1}{(x+1)^2}$

2. Find each limit.

a) $\lim_{x\to 1}\frac{x^2-1}{x-1}$ b) $\lim_{x\to -1^-}\frac{x-2}{x^2-x-2}$

c) $\lim_{x\to -1^+}\frac{x-2}{x^2-x-2}$ d) $\lim_{x\to -1}\frac{x-2}{x^2-x-2}$

e) $\lim_{x\to -2}\frac{x+1}{(x+2)^2}$ f) $\lim_{x\to -2}\frac{x^2-4}{(x+2)^2}$

3. Find the equation of any vertical asymptote.

a) $f(x)=\frac{1}{x}$ b) $f(x)=\frac{1}{x^2}$

c) $f(x)=\frac{1}{x-2}$ d) $f(x)=\frac{1}{(x+1)^2}$

e) $f(x)=\frac{x}{x-1}$ f) $f(x)=\frac{x-1}{x^2-1}$

g) $f(x)=\frac{x^2-1}{x-1}$ h) $f(x)=\frac{x+1}{x^2-x-2}$

i) $f(x)=\frac{x^2-9}{(x-3)^2}$ j) $f(x)=\frac{x^2-1}{x^4-1}$

k) $f(x)=\frac{1}{x^2-4}$ l) $f(x)=\frac{x+3}{x^2+x-2}$

4. Sketch the graph of the function near any vertical asymptote.

a) $f(x)=\frac{1}{x-2}$ b) $f(x)=\frac{1}{(x+1)^3}$

c) $f(x)=\frac{1}{x^2-4}$ d) $f(x)=\frac{x-2}{x^2+3x+2}$

e) $f(x)=\frac{x+1}{x^2+3x+2}$ f) $f(x)=\frac{x^2-4}{(x+2)^2}$

g) $f(x)=\frac{x^3-1}{x^2-1}$ h) $f(x)=\frac{x^3-1}{x^4-1}$

5. Find the limit and the equation of the horizontal asymptote, if they exist.

a) $\lim_{x\to\infty}\frac{1}{x}$ b) $\lim_{x\to-\infty}\frac{1}{x^2}$

c) $\lim_{x\to\infty}\frac{1}{\sqrt{x}}$ d) $\lim_{x\to-\infty}\frac{1}{\sqrt[3]{x}}$

e) $\lim_{x\to\infty}\frac{x}{x+1}$ f) $\lim_{x\to-\infty}\frac{x}{x^2+1}$

g) $\lim_{x\to\infty}\frac{1-x^2}{x^2+1}$ h) $\lim_{x\to-\infty}\frac{2x^2-3x+1}{-3x^2+x-2}$

i) $\lim_{x\to\infty}\sqrt{x}$ j) $\lim_{x\to\infty}\frac{x^2}{x+1}$

k) $\lim_{x\to\infty}\frac{-x^3-3x}{-2x^3+x^2}$ l) $\lim_{x\to-\infty}\frac{2x^4-1}{-x^4+1}$

6. Use the long division algorithm to find the equation of the oblique asymptote, if it exists.

a) $f(x)=\frac{x^2}{x+1}$ b) $f(x)=\frac{x^3+x}{x^2-1}$

c) $f(x)=\frac{x^2}{1-x}-\frac{x^2}{1+x}$ c) $f(x)=\frac{x^4-1}{x^3-1}$

7. Find each limit, if it exists. For each case, state the indeterminate form.

a) $\lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x})$ b) $\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{x+1}\right)$

c) $\lim_{x\to\infty}\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+1}}\right)$ d) $\lim_{x\to\infty}(\sqrt{2x}-\sqrt{x})$

e) $\lim_{x\to\infty}(\sqrt{x^2+1}-x)$ f) $\lim_{x\to\infty}(\sqrt{x^2+x}-x)$

g) $\lim_{x\to\infty}\frac{\sqrt{x}}{2\sqrt{x}+1}$ h) $\lim_{x\to\infty}\frac{\sqrt{x+1}}{\sqrt{4x-1}}$

i) $\lim_{x\to\infty}(\sqrt{x^2+x+1}-x)$ j) $\lim_{x\to\pm\infty}\frac{2x}{|x|+1}$

k) $\lim_{x\to\infty}\frac{\sqrt[3]{x}}{\sqrt{x^2+1}}$

8. Find the equation of any asymptote.

a) $f(x)=\frac{x+1}{x-1}$ b) $f(x)=\frac{x^2+1}{x^2-1}$

c) $f(x)=\frac{x^2+1}{x-2}$ d) $f(x)=\frac{x}{x^2-4}$

e) $f(x)=\frac{x|x|}{x+1}$ f) $f(x)=\frac{x^2-4}{|x|-2}$

Answers

1. a) $-\infty$ b) ∞ c) DNE d) ∞ e) ∞ f) ∞ g) $-\infty$ h) ∞
i) DNE j) ∞ k) ∞ l) ∞

2. a) 2 b) $-\infty$ c) ∞ d) DNE e) ∞ f) DNE

3. a) $x=0$ b) $x=0$ c) $x=2$ d) $x=-1$ e) $x=1$
f) $x=-1$ g) none h) $x=2$ i) $x=3$ j) none
k) $x=\pm 2$ l) $x=1$ and $x=-2$

5. a) $0, y=0$ b) $0, y=0$ c) $0, y=0$ d) $0, y=0$
e) $1, y=1$ f) $0, y=0$ g) $-1, y=-1$ h) $-2/3, y=-2/3$
i) ∞ j) ∞ k) $1/2, y=1/2$ l) $-2, y=-2$

6. a) $y=x-1$ b) $y=x$ c) $y=-2x$ d) $y=x$

7. a) 0 b) 0 c) 0 d) ∞ e) 0 f) $1/2$ g) $1/2$ h) $1/2$
i) $1/2$ j) $\lim_{x\to\infty} f(x)=2$ and $\lim_{x\to-\infty} f(x)=-2$ k) 0

8. a) HA: $y=1$, VA: $x=1$
b) HA: $y=1$, VA: $x=\pm 1$
c) VA: $x=2$, OA: $y=x+2$
d) VA: $x=\pm 2$, HA: $y=0$
e) VA: $x=-1$, OA: $y=x-1$ at ∞ and $y=-x+1$ at $-\infty$
f) OA: $y=x+2$ at ∞ and $y=-x+2$ at $-\infty$

