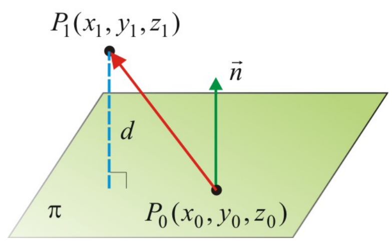
9.6 Distance from a Point to a Plane

A Distance from a Point to a Plane
Let consider a plane π with a *normal vector* $\vec{n}$ and a point $P_0(x_0, y_0, z_0)$ on this plane. The *distance* from a point $P_1(x_1, y_1, z_1)$ to the plane π is given by the *scalar projection* of the vector $\overrightarrow{P_0P_1}$ onto the normal vector $\vec{n}$:

$$d = \frac{|\overrightarrow{P_0P_1} \cdot \vec{n}|}{\|\vec{n}\|} \quad (1)$$



Ex 1. For each case, find the distance between the given plane and the given point.

a) $\vec{r} = (1,0,2) + t(0,1,2) + s(2,0,1)$, $B(2,3,0)$

$\overrightarrow{P_0B} = (2,3,0) - (1,0,2) = (1,3,-2)$

$$\begin{array}{lccccc} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \vec{n} = \vec{u} \times \vec{v} = & 0 & 1 & 2 & 0 & 1 \\ & 2 & 0 & 1 & 2 & 0 \end{array} = (1,4,-2)$$

$$\therefore d = \frac{|\overrightarrow{P_0B} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|1+12+4|}{\sqrt{1+16+4}} = \frac{17}{\sqrt{21}}$$

b) $\begin{cases} x = 1 - t + s \\ y = 2 - t - 2s \\ z = -1 + 2t - 3s \end{cases}$ $M(1,0,-2)$

$\overrightarrow{P_0M} = (1,0,-2) - (1,2,-1) = (0,-2,-1)$

$\vec{u} = (-1,-1,2); \quad \vec{v} = (1,-2,-3)$

$$\begin{array}{lccccc} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \vec{n} = \vec{u} \times \vec{v} = & -1 & -1 & 2 & -1 & -1 \\ & 1 & -2 & -3 & 1 & -2 \end{array}$$

$\vec{n} = (3+4, 2-3, 2+1) = (7,-1,3)$

$$\therefore d = \frac{|\overrightarrow{P_0M} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|0+2-3|}{\sqrt{49+1+9}} = \frac{1}{\sqrt{59}}$$

B Distance from a Point to a Plane (II)
If the plane π is given by the *Cartesian equation* $\pi : Ax + By + Cz + D = 0$, then the *distance* from a point $P_1(x_1, y_1, z_1)$ to the plane is given by:

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} \quad (2)$$

Indeed,
$P_0 \in \pi \Rightarrow Ax_0 + By_0 + Cz_0 + D = 0$
$\overrightarrow{P_0P_1} \cdot \vec{n} = (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot (A, B, C)$
$= Ax_1 + By_1 + Cz_1 - Ax_0 - By_0 - Cz_0 = Ax_1 + By_1 + Cz_1 + D$

Ex 2. Find the distance between the point $R(-2,0,3)$ and the plane $\pi : 2x - 3y + z - 6 = 0$.

$$\therefore d = \frac{|2(-2) - 3(0) + 3 - 6|}{\sqrt{2^2 + (-3)^2 + 1^2}} = \frac{|-4+3-6|}{\sqrt{4+9+1}} = \frac{7}{\sqrt{14}} = \frac{\sqrt{14}}{2}$$

C Distance between two Parallel Planes
To get the *distance* between *two parallel planes*:
a) Find a specific point into one of these planes.
b) Find the distance between that specific point and the other plane using one of the formulas above.

Ex 3. Find the distance between the parallel planes.
$\pi_1 : 3x + 6y - 9z - 3 = 0$, $\pi_2 : 2x + 4y - 6z - 4 = 0$

$P_1(1,0,0) \in \pi_1$

$$\therefore d = \frac{|2(1) + 4(0) - 6(0) - 4|}{\sqrt{4+16+36}} = \frac{2}{\sqrt{56}} = \frac{1}{\sqrt{14}}$$

Ex 4. Consider a plane π with the x-, y-, and z-intercepts equal to a, b, and c respectively.
a) Find the Cartesian equation of the plane π.

$$Ax+By+Cz+D=0$$

$$x-\text{int}=-\frac{D}{A}=a\Rightarrow A=\frac{-D}{a}$$

$$y-\text{int}=-\frac{D}{B}=b\Rightarrow B=\frac{-D}{b}$$

$$z-\text{int}=-\frac{D}{C}=c\Rightarrow C=\frac{-D}{c}$$

$$\frac{-D}{a}x+\frac{-D}{b}y+\frac{-D}{c}z+D=0$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0$$

b) Find the distance from the origin to the plane π.

$$d=\frac{|\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

Ex 5. Consider the plane $\pi:-x+2y-3z+10=0$ and the point $P(2,3,0)$.

a) Find the vector equation of the line $L_\perp$ that passes through the point P and is perpendicular to the plane π.

The line $L_\perp$ is perpendicular to the plane π and therefore is parallel to the normal vector $\vec{n}=(-1,2,-3)$ to the plane π. The normal vector $\vec{n}=(-1,2,-3)$ may be used as direction vector for the line $L_\perp$. The vector equation of the line $L_\perp$ is:

$$L_\perp:\vec{r}=(2,3,0)+t(-1,2,-3),\quad t\in R$$

b) Find the point of intersection F between the perpendicular line $L_\perp$ and the plane π.

$$L_\perp:\begin{cases}x=2-t\\y=3+2t\Rightarrow\pi\\z=-3t\end{cases}$$

$$-(2-t)+2(3+2t)-3(-3t)+10=0$$

$$14t=-14\Rightarrow t=-1\Rightarrow F:\begin{cases}x=2-(-1)=3\\y=3+2(-1)=1\\z=-3(-1)=3\end{cases}$$

$$\therefore F(3,1,3)$$

c) Find the shortest distance between the point P and the plane π.

$$d=\|\overrightarrow{PF}\|=\|(1,-2,3)\|=\sqrt{1+4+9}=\sqrt{14}$$

Reading: Nelson Textbook, Pages 542-549
Homework: Nelson Textbook: Page 549 # 1, 2, 3, 5, 6

