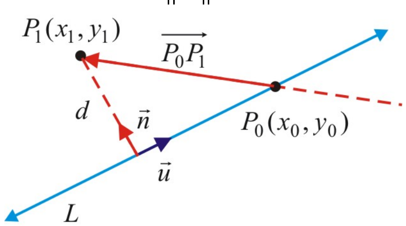
9.5 Distance from a Point to a Line

A Distance from a Point to a Line in R^2

Let $L: Ax + By + C = 0$ be a line in R^2, $P_1(x_1, y_1)$ be a *generic point* on the xy-plane and $P_0(x_0, y_0)$ be a *specific point* on this line, so: $Ax_0 + By_0 + C = 0$.

The *distance* d between the point $P_1(x_1, y_1)$ to the line L is given by (*scalar projection* of $\overrightarrow{P_0P_1}$ onto the normal vector $\vec{n}$):

$$d = \frac{|\overrightarrow{P_0P_1} \cdot \vec{n}|}{\|\vec{n}\|} \qquad (1)$$



Using $\vec{n} = (A, B)$, $\|\vec{n}\| = \sqrt{A^2 + B^2}$ and:

$$\overrightarrow{P_0P_1} \cdot \vec{n} = (x_1 - x_0, y_1 - y_0) \cdot (A, B)$$
$$= A(x_1 - x_0) + B(y_1 - y_0) = Ax_1 + By_1 - Ax_0 - By_0$$
$$= Ax_1 + By_1 + C$$

the formula (1) may be written as:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \qquad (2)$$

Ex 1. Find the distance between the point $P_1(3,-1)$ and the line $L: -2x + 3y + 6 = 0$.

Method #1 is based on formula (1).

If $x = 0$ then $y = -2$. So $P_0(0,-2) \in L$.

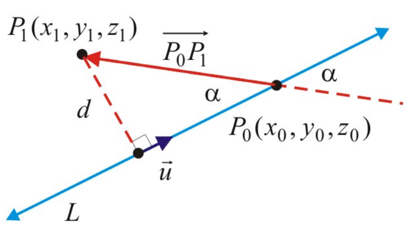
$$\overrightarrow{P_0P_1} = (3,-1) - (0,-2) = (3,1)$$
$$\vec{n} = (-2,3)$$
$$d = \frac{|\overrightarrow{P_0P_1} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|-6+3|}{\sqrt{4+9}} = \frac{3}{\sqrt{13}}$$

Method #2 is based on formula (2).

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|-2(3) + 3(-1) + 6|}{\sqrt{(-2)^2 + 3^2}} = \frac{3}{\sqrt{13}}$$

B Distance from a Point to a Line in R^3

Let $L: \vec{r} = \vec{r}_0 + t\vec{u}, \quad t \in R$ be a line defined by its vector equation and $P_0(x_0, y_0, z_0)$ be a specific point on this line.



The *distance* d from a point $P_1(x_1, y_1, z_1)$ to the line L may be found using:

$$d = \|\overrightarrow{P_0P_1}\| \sin\alpha \qquad (3)$$

Because $\|\overrightarrow{P_0P_1} \times \vec{u}\| = \|\overrightarrow{P_0P_1}\| \|\vec{u}\| \sin\alpha$, the formula (3) may be written as:

$$d = \frac{\|\overrightarrow{P_0P_1} \times \vec{u}\|}{\|\vec{u}\|} \qquad (4)$$

Note. The formula (4) may be applied also in R^2 by considering the third component $z = 0$.

Ex 2. For each case, find the distance from the given point to the given line.

a) $L: \vec{r} = (0,1,2) + t(-1,2,0), \quad t \in R; \quad P_1(1,2,-1)$

$$P_0(0,1,2), \quad P_1(1,2,-1) \Rightarrow \overrightarrow{P_0P_1} = (1,1,-3)$$

$$\begin{array}{lccccc} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \overrightarrow{P_0P_1} \times \vec{u} = & 1 & 1 & -3 & 1 & 1 \\ & -1 & 2 & 0 & -1 & 2 \end{array} = (6,3,3)$$

$$d = \frac{\|\overrightarrow{P_0P_1} \times \vec{u}\|}{\|\vec{u}\|} = \frac{\sqrt{36+9+9}}{\sqrt{1+4}} = \sqrt{\frac{54}{5}}$$

b) $L: \begin{cases} x = -1 + 2t \\ y = -3t \\ z = 2 \end{cases}, \quad t \in R; \quad O(0,0,0)$

$$P_0(-1,0,2), \quad O(0,0,0) \Rightarrow \overrightarrow{P_0O} = (1,0,-2); \quad \vec{u} = (2,-3,0)$$

$$\begin{array}{lccccc} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \overrightarrow{P_0O} \times \vec{u} = & 1 & 0 & -2 & 1 & 0 \\ & 2 & -3 & 0 & 2 & -3 \end{array} = (-6,-4,-3)$$

$$d = \frac{\|\overrightarrow{OP_0} \times \vec{u}\|}{\|\vec{u}\|} = \frac{\sqrt{36+16+9}}{\sqrt{4+9}} = \sqrt{\frac{61}{13}}$$

C Distance between two Parallel Lines

To find the *distance* between two parallel lines:

a) Find a *specific point* on one of these lines.

b) Find the distance from that specific point to the other line using one of the relations above.

Ex 3. Find the distance between the parallel lines: $L_1: 2x-3y+6=0$ and $L_2: -4x+6y-3=0$.

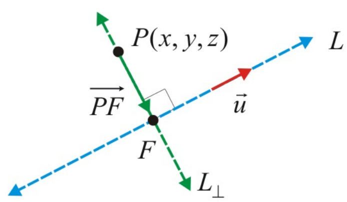
$P_1(0,2) \in L_1$

$$\therefore d = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}} = \frac{|-4(0)+6(2)-3|}{\sqrt{(-4)^2+6^2}} = \frac{9}{\sqrt{52}}$$

D Perpendicular Line from a Point to a Line

Let $L: \vec{r} = \vec{r}_0 + t\vec{u}, \quad t \in R$ be a line defined by its vector equation and $P(x,y,z)$ be a generic point in R^3.

The line perpendicular to the line L that passes through the point P is called the *perpendicular line* and intersects the line L at a point F called the *foot* of the perpendicular line.



The foot F of the perpendicular line may be found from the equation (because $\overrightarrow{PF} \perp \vec{u}$):

$$\overrightarrow{PF} \cdot \vec{u} = 0$$

A *vector equation* of the perpendicular line is:

$$\vec{r} = \overrightarrow{OP} + s\overrightarrow{PF}, \quad s \in R$$

Ex 4. Consider the line $L: \vec{r} = (-2,3,1) + t(1,-3,2), \quad t \in R$ and the point $P(3,-4,2)$.

a) Find the foot F of the perpendicular line $L_\perp$ from the point P to the line L.

$P(3,-4,2); \quad \vec{u} = (1,-3,2)$

$F(-2+t, 3-3t, 1+2t)$

$\overrightarrow{PF} = (-5+t, 7-3t, -1+2t)$

$\overrightarrow{PF} \cdot \vec{u} = 0 \Rightarrow -5+t-3(7-3t)+2(-1+2t) = 0$

$-5+t-21+9t-2+4t = 0 \Rightarrow 14t = 28 \Rightarrow t = 2$

$F(-2+2, 3-3(2), 1+2(2)) \Rightarrow \therefore F(0,-3,5)$

b) Find the equation of the perpendicular line $L_\perp$ from the point P to the line L.

$\overrightarrow{PF} = (0,-3,5) - (3,-4,2) = (-3,1,3)$

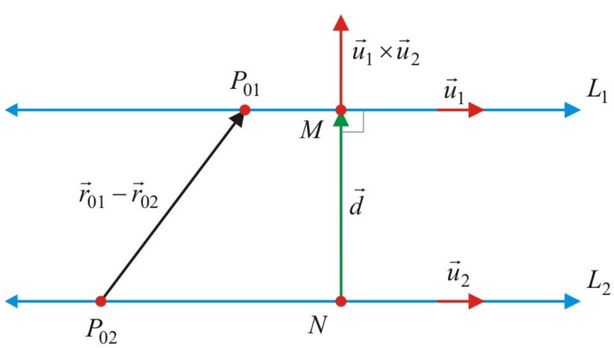
$\therefore L_\perp: \vec{r} = (3,-4,2) + s(-3,1,3), \quad s \in R$

c) Find the distance from the point P to the line L.

$\therefore d = \|\overrightarrow{PF}\| = \sqrt{9+1+9} = \sqrt{19}$

E Shortest Distance between two Skew Lines

Two skew lines lie into *two parallel planes*. The vector $\vec{u}_1 \times \vec{u}_2$ is *perpendicular* to both lines and therefore perpendicular to parallel plane the lines lie on. See the diagram below:



The *shortest distance* between two skew lines $L_1: \vec{r} = \vec{r}_{01} + t\vec{u}_1, \quad t \in R$ and $L_2: \vec{r} = \vec{r}_{02} + s\vec{u}_2, \quad s \in R$ is given by the *scalar projection* of the vector $\vec{r}_{01} - \vec{r}_{02}$ onto the vector $\vec{u}_1 \times \vec{u}_2$:

$$d = \frac{|(\vec{r}_{01} - \vec{r}_{02}) \cdot (\vec{u}_1 \times \vec{u}_2)|}{\|\vec{u}_1 \times \vec{u}_2\|}$$

Ex 5. Find the shortest distance between the lines $L_1: \vec{r} = (2,0,0) + t(1,1,0), \quad t \in R$ and $L_2: \vec{r} = (1,1,1) + s(0,-1,2), \quad s \in R$.

$\vec{r}_{01} = (2,0,0), \quad \vec{r}_{02} = (1,1,1) \quad \Rightarrow \vec{r}_{01} - \vec{r}_{02} = (1,-1,-1)$

$$\vec{u}_1 \times \vec{u}_2 = \begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 0 & -1 \end{array} = (2,-2,-1)$$

$$d = \frac{|(\vec{r}_{01} - \vec{r}_{02}) \cdot (\vec{u}_1 \times \vec{u}_2)|}{\|\vec{u}_1 \times \vec{u}_2\|} = \frac{|(1,-1,-1) \cdot (2,-2,-1)|}{\sqrt{4+4+1}}$$

$$= \frac{|2+2+1|}{\sqrt{9}} = \frac{5}{3}$$

$$\therefore d = \frac{5}{3}$$

Reading: Nelson Textbook, Pages 534-539

Homework: Nelson Textbook: Page 540 #1a, 2a, 3a, 5abc, 6a, 7a, 8, 10

